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STABILITY OF COUPLED-CORE NUCLEAR REACTOR SYSTEMS

by Hugh S. Murray and Lynn E. Weaver

Prepared under Grant No. NsG-490 *by*
UNIVERSITY OF ARIZONA
Tucson, Ariz.
for

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PREFACE

This report represents the completion of one phase of the study of coupled core reactor stability, a study sponsored by the National Aeronautics and Space Administration under Grant NsG-490 on research in and application of modern automatic control theory to nuclear rocket dynamics and control. The report is intended to be a self-contained unit and, therefore, repeats some of the work presented in previous status reports.

This work was submitted to the Department of Nuclear Engineering at The University of Arizona in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The type of coupled-core system with which this study is primarily concerned is that of two or more separate power reactors coupled together for the purpose of achieving an increased system power output. The primary example is that of the proposed clustering of nuclear rocket engines.

The behavior of a specific core in a coupled system is influenced by the past histories of all the other cores due to the transit time involved in the mutual exchange of leakage neutrons. The equations which describe the dynamics of a coupled-core system contain, because of this time lag in the interdependent portion of the system behavior, variables whose arguments are delayed or retarded in time. Herein lies the unique mathematical feature of the problem.

It is necessary from a practical standpoint to discover whether or not a projected coupled-core system is inherently stable. If, for example, a system of clustered nuclear rocket engines is operating at design power conditions, hopefully the system, if perturbed from the operating point, returns to its original state. This is the fundamental problem of the asymptotic stability of an autonomous or undriven system.

Recently, researchers have found that the most universal method of investigating the stability of ordinary differential equations is the Second Method of Liapunov. It is natural, therefore, to derive a technique based upon Liapunov's theory for this problem. The approach to

the stability problem via the Second Method has several advantages over the more conventional methods of analysis. The kinetics equations for power reactors are nonlinear due to temperature-induced reactivity effects. A conventional analysis might include a linearization of the describing equations and then perhaps a series approximation for the time delay. If series approximations are used for the delay, it is generally not possible to determine whether the results are conservative or falsely optimistic. Also, because the linearized equations are valid only in an arbitrarily small region of the variables about the operating point, it is not possible to determine the bounds on the initial conditions of the system within which the system is asymptotically stable.

The Second Method, on the other hand, is directly applicable to nonlinear equations. The presence of the time delay, however, necessitates some modifications to the Liapunov approach. The extension of the method is based upon the work performed by Driver, Krasovskii, and Razumikhin on the mathematical features of equations with delay. The time delay is incorporated directly into the Second Method, thus it is certain that the results are conservative due to the sufficiency of the stability conditions.

The modified approach is based upon comparing all possible solutions of a system with delay with the known properties of the system without delay. The technique used is to find quadratic Liapunov functions for the linearized, zero delay system. Stability conclusions can then be made for the linearized system with delay. Finally, the nonlinear terms are added to estimate the regions of stability of the state variables.

The results are adequate, from a practical viewpoint, for a simplified two-core example. However, the approach becomes highly restrictive for higher order, more complicated cases, and the results become marginal. The basic stability conclusions combined with simulation studies for the more complicated system, reveal that there are no practical stability problems involved in coupled-core nuclear reactor dynamics.

Chapter 1

INTRODUCTION

A coupled-core nuclear reactor is a critical reactor which consists of two or more independently subcritical cores. The multiplication of neutrons in each separate core is not sufficient to result in a self-supporting chain reaction. However, due to the leakage of neutrons from the physical boundaries of each core, there is a mutual exchange of neutrons among the cores. This is the neutronic coupling effect which balances the neutron economy of the entire reactor or system of cores such that a sustained nuclear fission reaction may occur.

Basically, there are three classes of systems which may be treated as coupled-core systems. The first class includes all large reactors whose properties of construction are uniform throughout the core, that is, the fuel-moderator arrangement is uniform. The coupled-core analysis in this case consists of subdividing the main core into several smaller cores and writing separate equations to describe the dynamics of each subdivision. The use of this technique leads to an approximate description of the spatial effects in the system behavior.

The second category of coupled-core reactor system comprises reactors which contain sections of completely distinct fuel-moderator material and geometric arrangement. In this case, it is necessary to describe each section by a different set of coupled equations. An example of this type of reactor is one which has two regions with different neutron energy spectra, fast and thermal, for the purpose of obtaining different heat transfer characteristics or to enhance fuel breeding in one region.

The third type of coupled-core system is the one with which this study is primarily concerned. This is the case of two or more separate power reactors coupled together for the purpose of achieving an increased system power output using an existing reactor system. The primary example is that of the proposed clustering of nuclear rocket engines (Seale, 1964a, 1964b). The individual cores in this arrangement are actually separate reactors designed and equipped to produce power by themselves.

Due to the coupling mechanism, the behavior of each core in the reactor is influenced by the behavior of all the other cores. In addition, a neutron in a given core, if it eventually is to cause fission in another core, requires a period of time to escape from the first core, travel between cores, enter the second core, and cause fission. Hence, the behavior of a specific core actually depends upon the past histories of all the other cores. The equations which describe the dynamics of a coupled-core system contain, due to this time lag in the interdependent portion of the system behavior, variables whose argument is delayed or retarded in time. Herein lies the unique mathematical feature of the problem.

A knowledge of the stability of a coupled-core reactor system is naturally quite important. It is possible in fact to raise a serious doubt as to whether or not a coupled system is stable using a simple physical argument. Suppose two reactors are operating together at the same power level. If the power of one reactor is increased, the power level of the other reactor also increases, thus affecting the originally perturbed reactor, and so on. At high power levels, of course, the increase in the core temperature leads to increased neutron leakage because of the decrease in core density. This intrinsic negative reactivity effect in general stabilizes an ordinary reactor.

It is necessary, from a practical standpoint, to discover whether or not a projected coupled-core system is inherently stable. If, for example, a system of clustered nuclear rocket engines is operating at design power conditions, hopefully the system, if perturbed from the operating point, returns to its original state. This is the fundamental problem of the asymptotic stability of an autonomous or undriven system. The purpose of this investigation is to solve this problem for coupled-core nuclear reactor systems.

Recently, researchers have found that the most universal method of investigating the stability of ordinary differential equations is the Second Method of Liapunov (LaSalle and Lefschetz, 1961). It is natural, therefore, to derive a technique based upon Liapunov's theory for this problem. The approach to the stability problem via the Second Method of Liapunov has several advantages over the more conventional methods of analysis. The kinetics equations for power reactors, such as a rocket system, are nonlinear due to the temperature-induced reactivity effects. The coupled-core equations also contain the time delay terms mentioned previously.

A conventional stability analysis might include a linearization of the describing equations, and then perhaps a truncated series approximation or Pade approximant (Weaver, 1963) to estimate the effect of the time delay. The exact solution for the roots of the characteristic equation of the linearized system is difficult because of the exponential time delay terms.

If series approximations are used for the time delay terms, it is generally not possible to determine whether the stability results are conservative or falsely optimistic. Also, because the linearized equations are valid only in an arbitrarily small region of the variables about the operating point, it is not possible to determine the bounds on the initial conditions of the system within which the system is asymptotically stable.

The Second Method of Liapunov, on the other hand, is directly applicable to nonlinear equations. The presence of the time delay, however, necessitates some modifications to the basic Liapunov approach. The Second Method yields only sufficient conditions for stability, so if the time delay feature is incorporated directly into the Second Method, it is certain that the results are either exact or conservative.

The reasons that the Second Method cannot be applied directly to the time delay problem will become apparent later. When the difficulties are overcome, the usefulness of the Second Method of Liapunov for solving the stability problem for coupled-core nuclear reactor systems will be demonstrated.

Chapter 2

THE KINETICS EQUATIONS FOR COUPLED-CORE REACTOR SYSTEMS

The model to be used in obtaining the system kinetics equations is that of a point reactor in which the various parameters and variables represent average values with respect to space. If the dynamic processes occurring within the system are understood, the lumped parameter model should yield a sufficiently accurate description of the behavior of the system.

General Neutron Kinetics

There is, in a given core, an average density n(t) of neutrons, all of which participate in the nuclear processes of the system at a single energy. Due to leakage from the system, parasitic absorption in the core materials, and productive absorption in the fuel, the rate of disappearance of these neutrons is $n(t)/\ell_o$, where ℓ_o is the mean lifetime of the neutrons. The productive absorption of neutrons in the fuel causes nuclear fission, accompanied by the release of energy and additional neutrons. Some of the produced neutrons do not appear at the instant of fission, but some time later. It is assumed that these delayed neutrons, which appear through the radioactive decay of unstable fission products, constitute a fraction β of the total fission neutrons. Although there are several distinct groups of nuclei which decay to produce the delayed neutrons, it is assumed that all the delayed neutrons can be attributed to one effective group of precursor atoms whose mean lifetime is $1/\lambda$ and whose density is C(t).

The effective multiplication of neutrons, or total number of neutrons produced in one generation per neutron in the previous generation is k . The delayed neutrons appear at the same rate at which the delayed neutron precursors decay, so the total rate of production of neutrons in the system is

$$\frac{k(1-\beta)}{\ell_o} n(t) + \lambda C(t)$$

where the first term is the prompt neutron source and the second term is the delayed neutron source.

The time rate of change of the neutron density is the difference between the rate of production and the rate of loss, or

$$\dot{n}(t) = \frac{\rho}{\ell} n(t) - \frac{\beta}{\ell} n(t) + \lambda C(t) + S(t) \qquad (2.1a)$$

$$\dot{C}(t) = \frac{\beta}{\ell} n(t) - \lambda C(t) \qquad (2.1b)$$

where Eq. (2.1b) describes the rate of change of the precursor atom density. The S(t) in Eq. (2.1a) is a general source term which accounts for additional internal or external neutron addition processes for a specific system. The parameter ℓ is $\frac{\ell_o}{k}$, the effective neutron lifetime. If the multiplication of the system is increased, a neutron is likely to survive over a greater effective length of time and, conversely, if the multiplication is decreased, the reaction goes more slowly and a neutron exists over a shorter period of time. Therefore, ℓ is assumed to be approximately constant.

The quantity ρ in Eq. (2.1a) is defined as the reactivity of the system, $(k-1)/k$, or the fractional change from unity of the effective multiplication factor.

The Delayed Source in Coupled Systems

The coupling effect in each core is assumed to appear in the form of a source due to neutrons coming from all ether cores in the system. Two assumptions are inherent in this formulation. First of all, it is assumed that for a given arrangement of cores, the rate of entry of neutrons into one core is directly proportional to the rate of loss of neutrons for the other cores. Secondly, it is assumed that the effective core-to-core transit time is the same on the average for all neutrons participating in the coupling process between two given cores. These premises are consistent with the lumped parameter model. Because all neutrons in a given core have the same energy and spatial distribution, they must exhibit the same average behavior in all nuclear processes.

If α_{ij} is the coupling coefficient of proportionality from the jth to the ith core, and if T_{ij} is the delay time associated with this coupling, the total source in the ith core of an N-core array is

$$S_i = \sum_{j=1 \neq i}^{N} \frac{\alpha_{ij}}{\ell_i} n_j(t-T_{ij}) \tag{2.2}$$

In general, $\alpha_{ij} = \alpha_{ji}$ and $T_{ij} = T_{ji}$, because the coupling effect is the same in either direction between any two cores. Also, all the α's and all the T's are equal in any completely symmetric array due to the fact that once the nuclear and neutronic properties of the system are fixed, the coupling process depends upon the geometric properties of the array. This is probably more true of α than of T. Geometric attenuation contributes almost exclusively to changes in α with separation if the neutron densities are the same in the cores. On the other hand, the time of flight of neutrons between cores is much less than the time required to escape from one core and enter another as there is no material between the cores. The coupling process is illustated in Fig. 2.1 for an asymmetric three-core array. In this case, all α's and all T's are equal except $\alpha_{13,31}$ is less than $\alpha_{12,21}$ or $\alpha_{23,32}$ and $T_{13,31}$ is greater than $T_{12,21}$ or $T_{23,32}$.

The basic neutron kinetics equations for coupled-core systems are, due to the substitution of Eq. (2.2) into Eq. (2.1),

$$\dot{n}_i(t) = \frac{\rho_i}{\ell} n_i(t) - \frac{\beta}{\ell} n_i(t) + \lambda C_i(t) + \sum_{j=1\neq i}^{N} \frac{\alpha_{ij}}{\ell} n_j(t-T_{ij}) \qquad (2.3a)$$

$$\dot{C}_i(t) = \frac{\beta}{\ell} n_i(t) - \lambda C_i(t) \qquad (2.3b)$$

The cores are assumed to have identical nuclear characteristics, so β, ℓ, and λ are the same throughout the system.

Equation (2.3) can be obtained from Eq. (2.1) also by means of a reactivity concept of the coupling effect. Seale (1964b) presents a computational method for obtaining theoretical values of α and T. The result is expressed as a change in k with respect to unity. This change in k in the ith core is shown to be proportional to the ratio

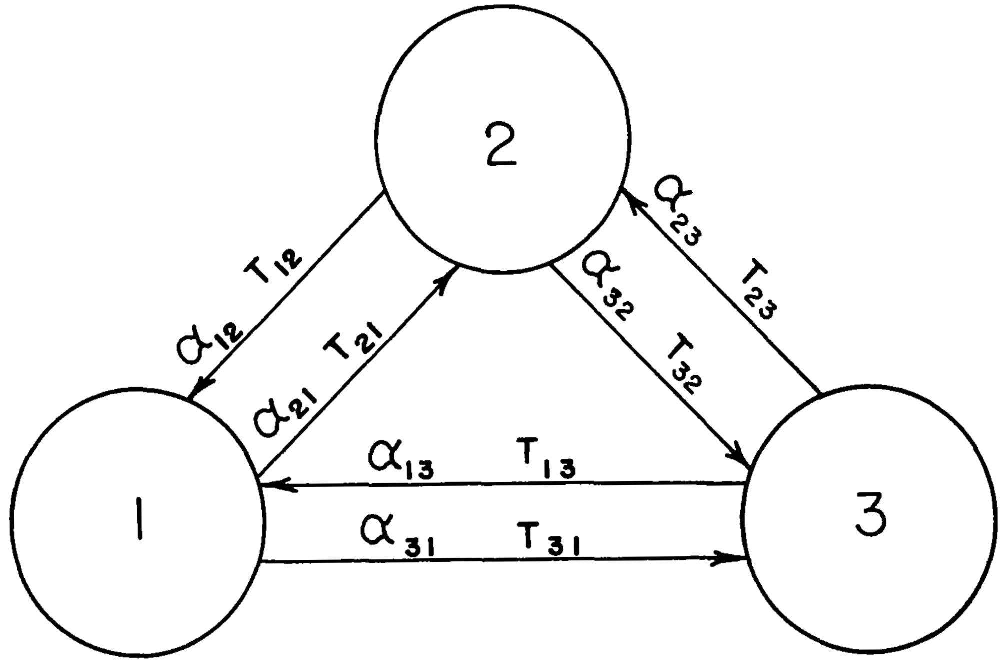


FIGURE 2.1.
NEUTRONIC COUPLING

of the surface neutron fluxes, hence the average neutron densities, in the ith and jth cores. The total coupling reactivity in the ith core is then

$$\rho_i = \sum_{j=1 \neq i}^{N} \alpha_{ij} \frac{n_j(t-T_{ij})}{n_i(t)} \qquad (2.4)$$

When the source term in Eq. (2.1) is eliminated and the reactivity replaced by (2.4) plus a generalized reactivity term, Eq. (2.3) are the result. The arguments are slightly different in concept but they yield the same system equations.

Higher order coupling effects are neglected in this study. Neutrons could escape from one core, reflect from one or more other cores, and return to the original core. It is highly doubtful that the magnitude of this self-coupling effect is appreciable due to at least one extra order of geometric attenuation.

System Heat Transfer Equations

Equations (2.3) represent the behavior in time of a core at very low power levels. In a power reactor, a coolant flows through the core to remove the generated energy. As the temperature of the core increases at higher power levels, the intrinsic reactivity effects appear. The density of the core decreases with an increase in temperature. As a result, it is less probable that a neutron will experience collisions with ether nuclei, including the fuel, and more probable that a neutron will escape from the system. Hence, the multiplication of neutrons decreases. This effect is the negative temperature reactivity effect.

Because a component of the reactivity is a function of temperature, it is necessary to describe the dynamic variations in the core temperature. The lumped parameter or point model for the heat transfer processes could lead to some difficulty because the temperatures and heat transfer parameters vary strongly within the core of a high-power reactor. It is of great importance to have proper effective values of these parameters to yield a fairly accurate model.

Over a period of time, the net accumulation of energy in a core is equal to the total energy generated due to fission minus the total energy removed by the coolant. Per unit time, the generated energy is the power P(t), which is proportional to the neutron density. The rate of energy removal is proportional to the difference between the average core temperature and the average coolant temperature T(t) and $T_c(t)$, respectively. These temperatures represent values relative to the zero power values. The energy balance for the core is

$$\int_0^t P(t)dt - \int_0^t H\left[T(t) - T_c(t)\right]dt = MC_r T(t)$$

which, when differentiated with respect to time, yields

$$MC_r \dot{T}(t) = P(t) - H\left[T(t)-T_c(t)\right] \quad . \qquad (2.5)$$

MC_r is the product of the mass and specific heat of the core, and H is the effective heat transfer coefficient between the core and the coolant.

A similar energy balance for the coolant yields

$$mC_c \dot{T}_c(t) = H\left[T(t)-T_c(t)\right] - wC_c T_c(t) \quad . \qquad (2.6)$$

The mass flow rate is w, and the mass heat capacity of the coolant is mC_c. It is assumed that the average core and coolant temperatures are proportional. If mC_c is very small, it can be seen from Eq. (2.6) that the assumption is valid. In a nuclear rocket, where gaseous hydrogen is the coolant, the quantity mC_c is small (Mohler, 1965). The mass flow rate of the coolant is constant because the system under consideration is autonomous. A combination of Eqs. (2.5) and (2.6) under these assumptions results in the core temperature kinetics equation,

$$\dot{T}(t) = \frac{1}{mC_r} P(t) - \omega T(t) \tag{2.7}$$

where ω is $\frac{HwC_c}{MC_r(H + wC_c)}$, the inverse of the characteristic heat exchanger time constant.

Intrinsic Reactivity Effects

The intrinsic reactivity effect is described under the assumption that an increase in core temperature causes a net decrease in neutron multiplication or the introduction of negative reactivity. It is supposed that this effect can be approximated by a linear function of temperature. The overall temperature reactivity effect is quite complicated due to changes in nuclear cross sections, material densities, and system dimensions, but generally the linear negative relationship provides an adequate description (Weaver, 1963, pp. 101-103).

A nuclear rocket is cooled by gaseous hydrogen, a strong neutron moderator, so temperature-induced coolant density changes affect the reactivity. The hydrogen density effect is a positive contribution, but it is roughly proportional to the inverse square root of the core temperature (Mohler, 1965). Therefore, the magnitude of the reactivity effect decreases with increasing temperature. Regardless of this positive intrinsic reactivity component, it will be supposed that the overall effect is negative and proportional to the temperature.

The temperature-dependent portion of the total reactivity is

$$\rho(T) = - C_T T(t) \quad , \tag{2.8}$$

where C_T is a positive constant of proportionality. If the fuel and moderator of the core constituted a homogeneous mixture, a prompt temperature effect model might be assumed. In this case, a change in power would result promptly in a corresponding change in temperature such that the intrinsic reactivity would be proportional to the power. Although this approach does not lead to a particularly accurate description of a practical system, it is useful for obtaining a rough prediction of the behavior of the system. Because the temperature kinetics equations are eliminated, the effort required to analyze such a model is minimized. The mathematical description of the prompt reactivity effect is

$$\rho(P) = -C_P P(t) \quad . \tag{2.9}$$

State Variable Representation of the System

The methods of treatment and the general discussions that follow presuppose that the nth order dynamic system is representable as n first order ordinary differential equations. Equations (2.3) and (2.7) are already first order differential equations. The time delay term causes no difficulty since ordinary differential equations are defined as equations containing the derivatives of the unknowns with respect to one real variable. The system equations therefore meet the above requirement as written.

Nevertheless, a further requirement is necessary. Questions of stability are considered with reference to some equilibrium point of the system. If the system unknowns are transformed such that they vanish at equilibrium, so do the derivatives of the variables. This defines the equilibrium point, which may be taken to be the origin of an n-dimensional Euclidean vector space. The state of the system is then completely described at any time by a vector which has as its components the new state variables. Stability may be discussed in terms of the region in this state space within which the system, if perturbed from the origin, returns to the origin. If the system exhibits this behavior, it is asymptotically stable.

The variables which translate the system operating point to a zero point are

$$x(t) = \frac{P(t)-P_o}{P_o}$$

$$y(t) = \frac{C(t)-C_o}{C_o} \qquad (2.10)$$

$$z(t) = \frac{T(t)-T_o}{T_o}$$

P_o is the equilibrium power, and C_o and T_o the equilibrium delayed neutron precursor density and average core temperature, respectively. The neutron density, it is recalled, may be written in terms of the power level.

The variables are normalized with respect to the operating level for convenience. One very good reason for normalizing is that the state variables now represent fractional changes in the actual system variables. Second and higher order terms in the state variables can be neglected directly if the linearized equations are desired.

An interrelation of the equilibrium values of the variables is found by setting the derivatives of Eqs. (2.3b) and (2.7) equal to zero. These relationships are

$$\frac{\beta}{\ell} P_{io} = \lambda C_{io} \tag{2.11}$$

$$\frac{1}{mC_r} P_{io} = \omega T_{io} \quad .$$

The total reactivity in the absence of external inputs is

$$\rho_i = \rho_{ioo} + \rho(T,P)$$

where ρ_{ioo} is the reactivity required to maintain criticality in the ith core at the operating point, and $\rho(T,P)$ is the intrinsic reactivity effect, a function of T or P . Under the transformations (2.10), the reactivity is

$$\rho_i = \rho_{ioo} + \delta_{io} + \delta_i(x,z) \tag{2.12}$$

where δ_{io} the intrinsic reactivity contributed at the operating point, and $\delta_i(x,z)$ is the component of the intrinsic reactivity which vanishes at the operating point.

The system equations for the ith core are, from the transformations (2.10),

$$\dot{x}_i(t) = \frac{\delta_i(x,z)}{\ell}\left[1+x_i(t)\right] - \frac{\beta}{\ell}x_i(t) + \frac{\beta}{\ell}y_i(t) \qquad (2.13a)$$

$$- \sum_{j=1\neq i}^{N} \frac{P_{jo}}{P_{io}}\frac{\alpha_{ij}}{\ell}x_i(t) + \sum_{j=1\neq i}^{N}\frac{P_{jo}}{P_{io}}\frac{\alpha_{ij}}{\ell}x_j(t-T_{ij})$$

$$\dot{y}_i(t) = \lambda x_i(t) - \lambda y_i(t) \qquad (2.13b)$$

$$\dot{z}_i(t) = \omega x_i(t) - \omega z_i(t) \qquad (2.13c)$$

and the value of ρ_{ioo} is

$$\rho_{ioo} = - \delta_{io} - \sum_{j=1\neq i}^{N} \alpha_{ij}\frac{P_{jo}}{P_{io}} \quad .$$

Note that the coupling terms contain the ratios of the power levels in the jth cores to that in the ith core. Clearly, the system can operate in equilibrium with all of the cores operating at different power levels.

The reactivity component δ_{io} vanishes at zero power, so ρ_{ioo} is negative at zero power. This verifies the individual subcriticality of the cores at this point.

The specific intrinsic reactivity components are, for the case in which temperature and power are proportional,

$$\begin{aligned} \delta_{io} &= - CpP_{io} \\ \delta_i(x) &= -CpP_{io}x_i(t) \end{aligned} \qquad (2.14)$$

and, for the actual temperature dependent case,

$$\delta_{io} = -C_T \, T_{io}$$

$$\delta_i(z) = -C_T \, T_{io} z_i(t) \quad . \qquad (2.15)$$

Equations (2.13) meet the requirements for the state variable notation. The derivatives vanish at the null point of all the variables. Also, the derivatives vanish when $x = y = z = -1$. This is to be expected since this point is zero power, or the case when the system is completely shut down. It is obvious that Eq. (2.13a) is nonlinear due to the term

$$\frac{\delta_i(x,z)}{\ell} \left[1 + x_i(t) \right] \quad . \qquad (2.16)$$

The equation is quickly linearized by writing the term (2.16) as

$$\frac{\delta_i(x,z)}{\ell} \quad .$$

Although this nonlinearity leads to some difficulty, it cannot be stressed too heavily that the most peculiar aspect of this problem is the presence in Eq. (2.13a) of the unknown with the retarded argument.

Vector-Matrix and Functional Notation

There is no point at this stage of the development in discussing a specific problem using the vector notation. There are, however, some general notational problems. Consider first the case in which there is no time delay. Ordinarily, all the equations are written in terms of one subscripted unknown, for example, x_i . In this case, the vectorial description of the system is

$$\underline{\dot{x}} = \underline{A}\ (\underline{x})\ \underline{x}$$

where $\underline{\dot{x}}$ and $\underline{x}$ are column vectors and A(x) is a square matrix, shown to be nonlinear in general.

For the coupled-core problem, three unknowns, x, y, and z are used in order that the subscripts be reserved to denote the specific core under consideration. The notation is

$$\left[\underline{\dot{x},\dot{y},\dot{z}}\right] = \underline{A}\ (\underline{x},\underline{z})\ \left[\underline{x,y,z}\right] .$$

The equations with time delay can be written in the completely general functional form

$$\left[\underline{\dot{z},\dot{y},\dot{z}}\right] = \underline{\mathfrak{F}}\ \left[\underline{x(s),y(t),z(t)}\right]$$

where the variable s includes all arguments $t - T \leq \underline{S} \leq t$, where T is the maximum delay that appears in the equations. $\underline{\mathfrak{F}}$ is a general functional of the variables with their various arguments.

Some Numerical Values for the System Parameters

The numerical values of the parameters listed below are based upon those used by Mohler (1965) in studying nuclear rocket systems. The values used here are rounded off to some degree, but represent, for the most part, typical parameters.

P_o design = 2000 MW

$\beta = 6 \times 10^{-3}$

$\ell = 4 \times 10^{-5}$ sec

$\lambda = 0.1\ \text{sec}^{-1}$

$$\frac{1}{\omega} = 1.5 \text{ sec}$$

$$C_P P_O \text{ or } C_T T_O = \beta \text{ at } 2000 \text{ MW} .$$

The coupling parameters are calculated by Seale (1964b) for ROVER type rocket cores. At a separation distance of fifteen feet for two such cores,

$$\alpha = 6 \times 10^{-4}$$

$$T = 3 \times 10^{-4} \text{ sec} .$$

The coupling parameters could be extrapolated to different separation distances by assuming the α varies geometrically, or inversely with the square of the separation distance, and that T is proportional to the separation. The general approach to the problem will be to seek the effect of variations in α and T on the system stability.

Chapter 3

STABILITY AND THE SECOND METHOD OF LIAPUNOV FOR TIME DELAY SYSTEMS

This chapter deals with the novel aspects of the stability problem for time delay systems. The primary goal of course is to adapt the Second Method of Liapunov to systems with time delay. It is necessary first, however, to discuss the basic concepts of the Second Method for ordinary systems. The reasons for the inadequacy of the approach as stated become apparent.

The Second Method of Liapunov for Ordinary Systems

The idea of the Second or Direct Method of Liapunov is to determine stability for a system without an actual knowledge of the solutions of the system. The solutions to nonlinear systems are not generally available; consequently, the Second Method is especially appropriate for studying nonlinear stability. The tool for solving the stability problem is the Liapunov function $V(\underline{x})$, a scalar function of the vector $\underline{x}(t)$ for the general nonlinear autonomous system

$$\dot{\underline{x}} = \underline{A}\ (\underline{x})\ \underline{x} \tag{3.1}$$

For simplicity, $V(\underline{x})$ will be denoted as V hereafter.

The function V, if it is to be a Liapunov function, is a positive-definite function according to the following definition (LaSalle and Lefschetz, 1961).

Definition 3.1 -- Liapunov Function

If a function V is

(a) continuous with continuous first partial derivatives in region H about the origin of the state space,

(b) positive in H except at the origin, where $V(0) = 0$, (V is positive-definite),

then V is a Liapunov function.

The theorem for asymptotic stability is (LaSalle and Lefschetz, 1961).

Theorem 3.1 -- Asymptotic Stability

If there exists in some region H about the origin of the state space, a Liapunov function, V, and if $\dot{V}$, the derivative with respect to time, is negative-definite along solutions of Eq. (3.1) in H, the origin is asymptotically stable.

The physical interpretation of Theorem 3.1 is illustrated in Fig. 3.1. The equation V = const. represents a series of closed surfaces about the origin. If $\dot{V}$ for the system is negative-definite, the state of the system for all initial conditions enclosed by H lies on successively smaller V's toward the origin. The region H for a linear system is infinite in extent and the stability is global.

The conditions for asymptotic stability by the Second Method are only sufficient. Hence, the fact that stability cannot be proved does not lead to the conclusion that the system is unstable. The problem is to select a suitable V such that asymptotic stability can be discovered if the system is asymptotically stable. There are obviously

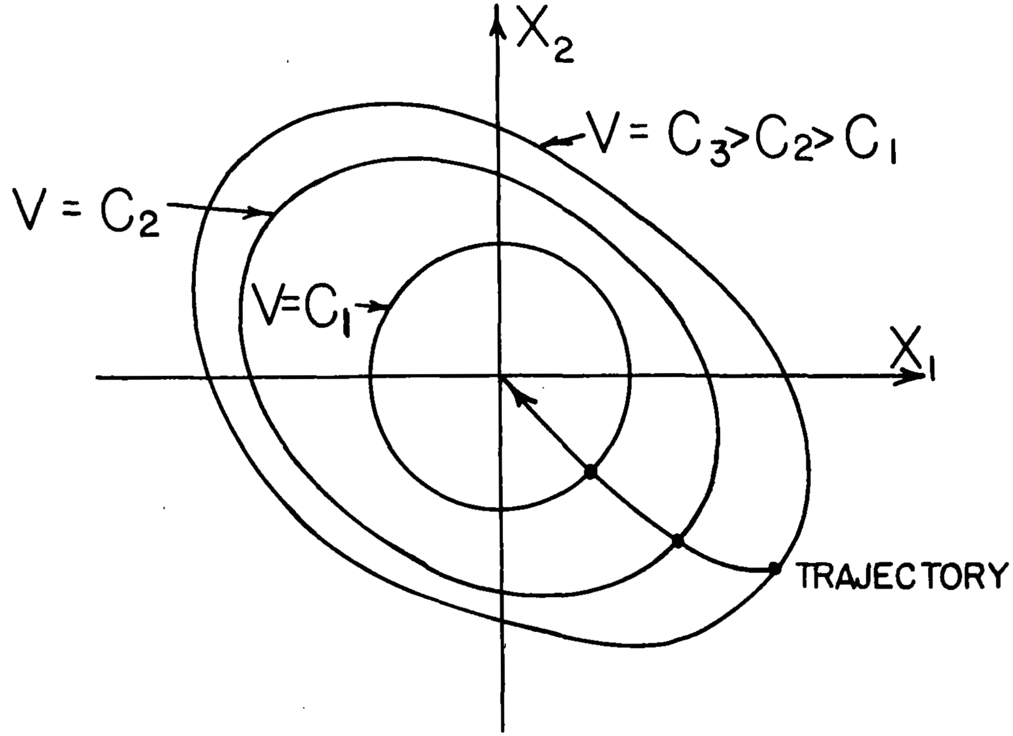


FIGURE 3.1.

ASYMPTOTIC STABILITY

a great many V's which meet the conditions of Definition 3.1 and Theorem 3.1 for any stable system.

An important V, especially for this problem, is the general quadratic form

$$V = \sum_{i,j=1}^{n} b_{ij} x_i x_j \tag{3.2}$$

Sylvester's Theorem (LaSalle and Lefschetz, 1961) gives the conditions for the sign-definiteness of a quadratic form. This will prove extremely useful.

Theorem 3.2 -- Sign-Definiteness of a Quadratic Form

The function $\sum_{i,j=1}^{n} b_{ij} x_i x_j$ is positive(negative)-definite if the successive principal minors of the symmetric determinant $|b_{ij}|$ are positive(negative).

The second order quadratic form

$$V = b_{11} x_1^2 + 2 b_{12} x_1 x_2 + b_{22}\, x_2^2$$

is positive definite by Theorem 3.2 if

$$b_{11} > 0$$

$$b_{11} b_{12} - b_{12}^2 > 0$$

The general matrix notation for the quadratic form is

$$V = \underline{x}^T\ \underline{B}\ \underline{x} \tag{3.3}$$

where $\underline{B}$ is the symmetric matrix

$$\underline{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

and $\underline{x}^T$ is the transpose of the column state variable vector $\underline{x}$.

The mechanics of solving the stability problem using a quadratic form include a linearization of the Eqs. (3.1). $\dot{V}$ is then constrained to be negative-definite along solutions of the linear system. This process determines the admissible values of the b_{ij}'s. The next step is to calculate $\dot{V}$ for the nonlinear system, which is, from Eq. (3.1) and (3.3),

$$\dot{V} = \underline{x}^T \underline{B} \left[\underline{A}(\underline{x})\underline{x}\right] + \left[\underline{A}(\underline{x})\underline{x}\right]^T \underline{B} \underline{x} \qquad (3.4)$$

Eq. (3.4) is then examined to find the values of x for which $\dot{V}$ is negative. The region H in the state space is the interior of the largest V which fits in the region $\dot{V} < 0$.

A method whereby a quadratic V is essentially generated is found by diagonalizing the linearized Eqs. (3.1) and describing the system in its canonic variables, Z. Eq. (3.1) is separated into a linear plus a nonlinear component,

$$\underline{\dot{x}} = \underline{A} \underline{x} + \underline{I} \underline{F} (\underline{x}) \qquad (3.5)$$

with $\underline{I}$ the square unit matrix, and $\underline{F}(\underline{x})$ the column vector function of the assumed separable nonlinearities. The solutions to the linear system take the form

$$\exp (\psi_i t)$$

where the ψ_i's are the eigenvalues which satisfy the determinantal equation

$$|\underline{A} - \psi \underline{I}| = 0 \ .$$

If all the eigenvalues have negative real parts, the solutions decrease exponentially, therefore the system is asymptotically stable.

The linear transformation

$$\underline{x} = \underline{P}\ \underline{Z} \tag{3.6}$$

results in a linear system representation

$$\underline{Z} = \underline{P}^{-1}\ \underline{A}\ \underline{P}\ \underline{Z} \ .$$

Furthermore, if

$$\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{\Lambda}$$

where $\underline{\Lambda}$ is a diagonal matrix with the eigenvalues as its elements, then the variables Z are the canonic variables of the linear system. The V function

$$V = \underline{Z}^T\ \underline{I}\ \underline{Z} = \sum_{i=1}^{n} Z_i^{\,2} \quad ,$$

when differentiated with respect to time, yields

$$\dot{V} = 2\ \underline{Z}^T\ \underline{\Lambda}\ \underline{Z} = 2 \sum_{i=1}^{n} \psi_i\ Z_i^{\,2} \quad .$$

Clearly, V is positive-definite and since the elements of $\underline{\Lambda}$ are negative by the stability requirements of the linear system $\dot{V}$ is negative-definite. Thus asymptotic stability is concluded. The final steps are to transform back to the real system variables x by means of Eq. (3.6), add the nonlinear terms to $\dot{V}$, and find the region H .

This approach is at least methodical. However, considerable difficulty in the mechanics of the method can be envisioned. The labor involved in finding the eigenvalues and the $\underline{P}$ matrix along with its inverse, would be prohibitive for high order systems. This approach will be useful as a guide toward drawing conclusions on the general properties of quadratic V functions for the practical problems to be considered.

A non-quadratic V function could be sought for the general nonlinear system. Specifically, the Variable Gradient Method (Schultz, 1962) assumes a general gradient of V from which $\dot{V}$ is found to be

$$\dot{V} = \underline{\nabla V}^T \, \underline{\dot{x}} \quad .$$

$\dot{V}$ is constrained to be negative-definite by a proper choice of the coefficients in $\underline{\nabla V}$. Since these coefficients may be functions of the state variables, a line integration of $\underline{\nabla V}$ may yield a non-quadratic or even a non-algebraic V . Although this method may yield excellent results ordinarily, it will be seen that only quadratic V functions can be easily extended to include time delay problems.

An Attempt to Extend the Second Method to Time Delay Systems

In order to illustrate dramatically the difficulties involved in this problem, an attempt is made to extend the Second Method directly to the first order time delay system

$$\underline{\dot{x}}(t) = - ax(t) - bx(t-T) \quad . \qquad (3.7)$$

The quadratic V is

$$V = x^2(t)$$

and the time derivative is

$$\frac{1}{2}\dot{V} = -a\, x^2(t) - bx(t)\; x\;(t-T) \qquad (38)$$

The method fails. Due to the cross product $x(t)x(t-T)$, no conclusion as to the sign-definiteness of V can be drawn.

Some insight can be obtained by actually solving Eq. (3.7). It is immediately apparent that a solution for $t \geq 0$ depends upon known values of x for all $-T \leq t \leq 0$. This is an important difference between systems with and without time delay. The solutions to differential equations without the retarded argument depend only upon the initial conditions or values of the variables at $t = 0$. The solutions to time delay equations, however, depend upon entire initial functions defined over $-T \leq t \leq 0$. Thus the solutions to two time delay equations could have the same value at the initial instant. $t = 0$, but describe completely different trajectories due to different initial functions.

The equation (3.7) is solved by step-wise integration over each interval of time of length T starting at $t = 0$. If $x = 1$ for $-T \leq t \leq 0$, the result is

$$x_n(t) = (-\frac{b}{a})^n$$

$$+ (1+\frac{b}{a}) \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{(-1)^{k-1}\left[t-(k-1)T\right]^{j-1}(b)^{j-1}(\frac{b}{a})^{k-j}\exp\left\{-a\left[t-(k-1)T\right]\right\}}{(j-1)!}$$

where $x_n(t)$ is the solution for the time interval $(n-1)T \leq t \leq nT$.

The general form shown is obtained by induction from the solutions for a few time intervals.

The solution of Eq. (3.7) for $a=b=T=1$ is shown in Fig. (3.2). Because this is a first order system, the oscillatory behavior is most unusual. First order systems with no time delay exhibit purely real exponential behavior. Only second or greater order systems with no delay may oscillate.

Because x oscillates, $V = x^2$ also oscillates. Therefore, $\dot{V}$ must be positive part of the time, even though Fig. (3.2) suggests an asymptotically stable behavior. The specific solution in time of V for this example appears in Fig. (3.3).

Modification of the Second Method for Time Delay Systems

The Second Method of Liapunov must be generalized to include cases such as the one examined in the previous section. In reality, this does not lead to a modification of the fundamental concepts of the Second Method. The idea still is to find a positive-definite Liapunov function and to draw asymptotic stability conclusions based upon the negative-definiteness of its time derivative. The changes are made within the asymptotic stability theorem. The V function for an asymptotically stable system with delay must eventually decrease toward the origin of the state space, even though it does not demonstrate a monotonic behavior. By way of comparison with systems with no delay, consider an ordinary damped spring-mass system. A suitable V for this case is the kinetic plus the potential energy of the mass. Even though the system state variables, position and velocity, oscillate, the total

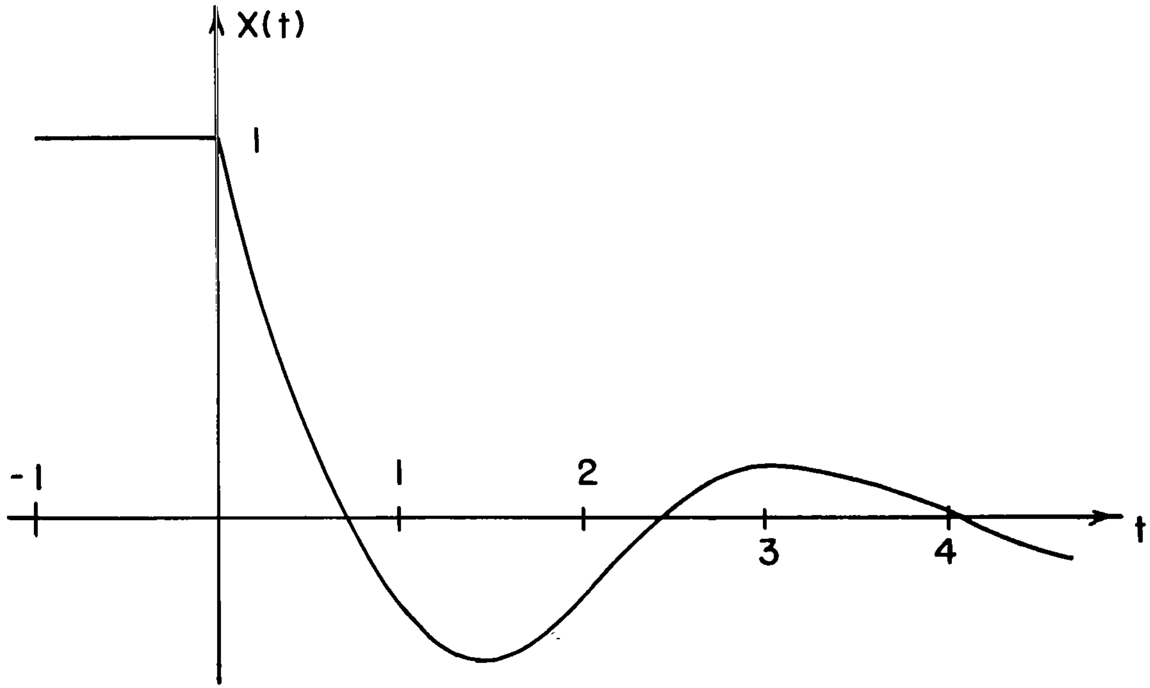


FIGURE 3.2.

SOLUTION OF $\dot{X}(t) = -X(t) - X(t-1)$

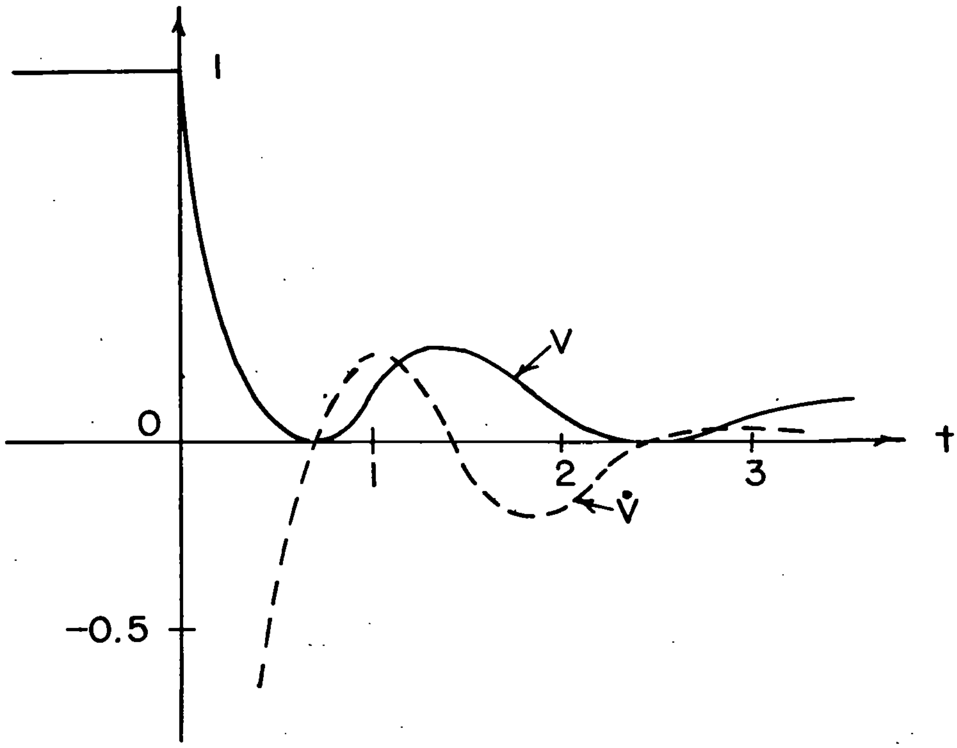


FIGURE 3.3.

$$V = X^2(t)$$

FOR

$$\dot{X}(t) = -X(t) - X(t-1)$$

energy or V decreases monotonically for all time.

Before asymptotic stability is considered, it is instructive to discuss ordinary stability or boundedness of time delay systems. This approach provides some insight to the difficulties involved. Driver (1962, p. 405) states an important lemma dealing with the relationship between time delay systems and ordinary systems.

Lemma 3.1

If there exists a continuous, non-negative function $\omega(r)$ for $t \geq 0$, $r \geq 0$, and a continuous, non-negative function $V(t)$ for $t \geq - T$ and if:

(a) $\dot{V}(t) \leq \omega\,[V(t)]$ whenever $V(S) \leq V(t)$ for all $t-T \leq S \leq t$;

(b) $r_o \geq \sup V(s)$ for $-T \leq S \leq 0$;

(c) the solution $r(t)$ of $\dot{r}(t) = \omega[r(t)]$, $r(0) = r_o$, exists for $t \geq 0$;

then

$$V(t) \leq r(t)$$

for all $t \geq 0$.

The lemma is proved by Driver (1962, pp. 405-406). Lemma 3.1 states that if the derivative of V is bounded by ω whenever V is growing, then all solutions V are bounded by the function r. This is shown in Fig. 3.4.

Driver's stability theorem is (1962, p. 417) now stated.

Theorem 3.3 - Stability of Time Delay Systems

If the function $\omega(t,r)$ is non-negative and if there exists a function $V(t,x)$ defined whenever $t > -T$ and $\|x\| < H$ in E

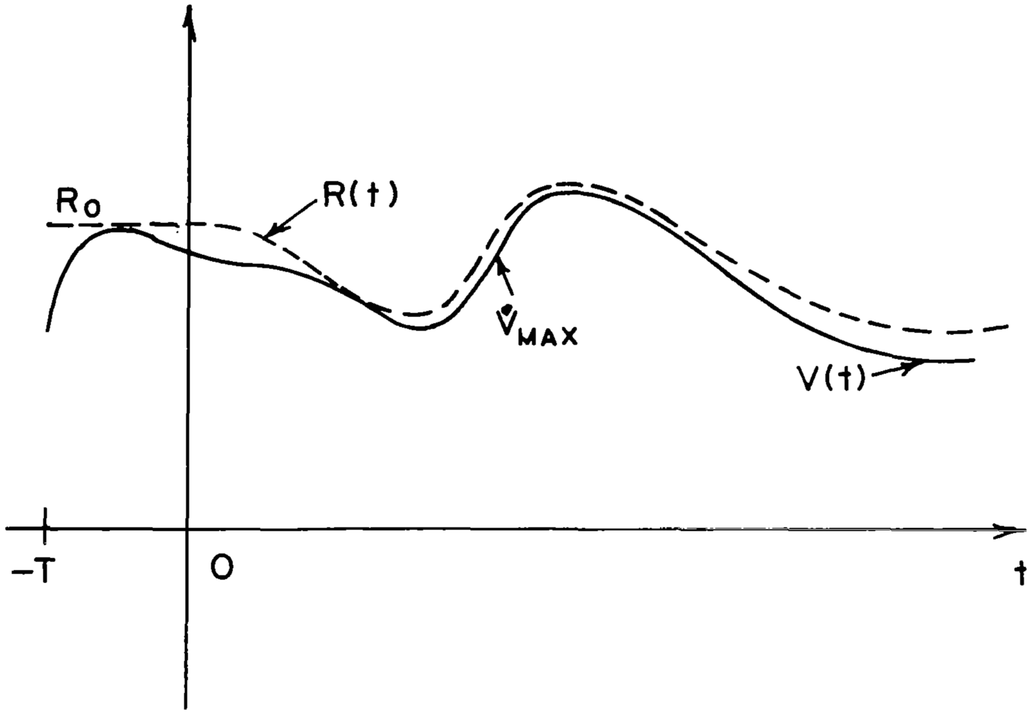


FIGURE 3.4.

RELATION BETWEEN ORDINARY AND DELAY SOLUTIONS such that

(a) $V(t,o) = 0$;

(b) $V(t,x)$ is continuous in t and locally Lipschitz with respect to x (for all $0 \leq \gamma < \infty$ and every compact set F in H there exists a constant $L_{\gamma,F}$ such that

$$|V(t,x_1) - V(t,x_2)| \leq L_{\gamma,F} \|x_1 - x_2\|$$

whenever $-T \leq t \leq \gamma$ and x_1 , x_2 in F) ;

(c) $V(t,x) \geq w_1(x)$, a positive continuous function for all $\|x\| < H$ except at the origin of E, and

(d) defining $\mathcal{V}(t,x(s)) = V(t,x(t))$, we have

$$\dot{\mathcal{V}}_*(t,x(s)) \leq \omega[t,V(t,x(t))]$$

whenever $t > 0$ and x continuous in H from $-T$ to t, and

(whenever) $\quad V[s,x(s)] \leq V[t,x(t)]$

for all $t - T \leq s \leq t$,

then the solution $x(t) = 0$ of (*) is stable to the right of $t = 0$.

This theorem is quoted from Driver's paper except that he does not use the symbol x for his system dependent variable and he considers time-dependent delays instead of the constant T. The notation (*) refers to the system equation

$$\dot{\underline{x}}(t) = \underline{\mathcal{F}}[\underline{x}(s)]$$

and, accordingly, $\dot{V}_*$ is the derivative of V along solutions of the system.

Theorem 3.3 also considers the non-autonomous or time-varying cases. In this study, such systems are of no interest. Hence, condition (c) merely means that V is positive-definite; then condition (a) is always satisfied.

The notation $\|\underline{x}\|$ refers to the norm of $\underline{x}$ or the maximum length in the region H. These new notational features will be used henceforth.

From Lemma 3.1 and Theorem 3.3, it is apparent that the only useful $\omega(V)$ is $\omega = 0$. In this case, all solutions to the delayed system are bounded by a constant. Equation (3.7) is considered in order to illustrate the application of Theorem 3.3 to time delay problems. V is chosen to be

$$V = x^2 ,$$

and along solutions of (3.7),

$$\frac{1}{2}\dot{V} = -ax^2(t) - bx(t)x(t-T) .$$

Whenever $$V\left[x(s)\right] \leq V\left[x(t)\right] ,$$

$$x^2(s) \leq x^2(t) ;$$

hence

$$|x(s)| \leq |x(t)|$$

for all $t-T \leq S \leq t$. It is clear that

$$\frac{1}{2}\dot{V} \leq -ax^2(t) + |bx(t)x(t-T)|$$

and whenever $$V\left[x(s)\right] \leq V\left[x(t)\right] ,$$

$$\frac{1}{2}\dot{V} \leq -\left[a - |b|\right] x^2(t) .$$

Because $$V = x^2(t) ,$$

$$\omega(V) = -2\left[a - |b|\right] V ,$$

thus, by Lemma 3.1, all solutions V are bounded by the maximal solution of

$$\dot{V} = 0$$

if $a = |b|$, or $\omega(V) = 0$. The solution of $\dot{V} = 0$ is

$$V = \sup x^2(s) \; ; \qquad -T \leq S \leq 0 \, ,$$

so all solutions to Eq. (3.7) are contained in the region

$$|x(s)| < \sup |x(s)| \; ; \; -T \leq S \leq 0$$

or within the region bounded by the maximum value of the initial conditions of Eq. (3.7).

The conclusion is that the solutions of the system are stable under the conditions given independent of the delay time T. In the general case, the variable s describes all values of the delay from zero up to some maximum value T if several variables are delayed. Again, this theorem concludes stability but not asymptotic stability, although there may be solutions in the region $V \leq r_0$ which are asymptotically stable.

It appears on the basis of Theorem 3.3 that asymptotic stability is difficult to show. Driver's asymptotic stability theorem (1962, p. 422), however, makes this task less difficult.

<u>Theorem 3.4</u> - <u>Asymptotic Stability of Time Delay Systems</u>

Consider equation (*) with the assumption that $\mathcal{F}(t,x(s)) = \mathcal{F}(t,x(s),g(t))$ where $g(t) \to \infty$ as $t \to \infty$. If there exists a function $V(t,x)$ defined whenever $t \geq g(0)$ and x in H such that

(a) $V(t,x) \leq W(x)$, a continuous function for all x in H with $W(0) = 0$ (an infinitesimal upper bound);

(b) $V(t,x)$ is continuous in t and locally Lipschitz with respect to x (as defined in Theorem 3.3);

(c) $V(t,x) \geq w_1(x)$, a positive continuous function for all x in H except at the origin of E, and

(d) There exists a continuous, non-decreasing function $f(r) > r$ for all $r > 0$ and a continuous function $w(x) > 0$ for all x in H except at the origin of E such that

$$\dot{V}_*[t,x(s),g(t)] \leq - w\,[x(t)]$$

whenever $t \geq 0$, x continuous in H for $g(t) \leq s \leq t$, and (whenever)

$$V[s,x(s)] < f\left\{V[t,x(t)]\right\}$$

for all $g(t) \leq s \leq t$,

then the solution $x(t) = 0$ of (*) is uniformly stable and asymptotically stable to the right of $t = 0$. If $g(t) \geq t - T$ for $t \geq 0$, where $T \geq 0$ is a constant, then the asymptotic stability is uniform.

The theorem as quoted is proved by Driver (1962, pp. 422-424). In Theorem 3.4, only bounded types of delays are considered. In this study, specifically, $g(t) = t - T$. As before, because only the time-invariant case is of interest, conditions (a) and (c) merely assert the positive-definiteness of V. When this is true, the functions $W(x)$ and $w_1(x)$ and identical to $V(x)$.

The important condition of Theorem 3.4 is condition (d). This condition requires that $\dot{V}$ is negative-definite whenever $f\{V[x(t)]\}$ is greater than V has been in the past, back to $t - T$. This condition admits V's which decrease nonmonotonically. Examples of stable and unstable behavior are given in Fig. 3.5 for $f(V) = V/q$ with $0 < q < 1$.

Krasovskii (1963, p. 157) gives an asymptotic stability theorem which is incorrectly stated (Driver, 1965). Krasovskii appears to require that the delay inequalities $V[x(x)] < f\{V[x(t)]\}$ hold as a condition of the theorem. That this is not true can be seen from the example figures. Theorem 3.4 retains the generality of a theorem for systems with no delay. In this event, the delay inequalities are not needed to show the negative-definiteness of $\dot{V}$ because V has a monotonic behavior (Driver, 1965).

As an example, again Eq. (3.7) is considered. If

$$V = x^2$$

then

$$\frac{1}{2}\dot{V} = -ax^2(t) - bx(t)x(t-T).$$

Whenever

$$V[x(s)] < \frac{1}{q} V[x(t)] , \qquad (3.9)$$

$$|x(s)| < \frac{1}{\sqrt{q}} |x(t)| .$$

So $-w(x)$ is

$$-w(x) = -(a - \frac{|b|}{\sqrt{q}})x^2(t) , \qquad (3.10)$$

which is negative-definite for

$$a\sqrt{q} > |b| . \qquad (3.11)$$

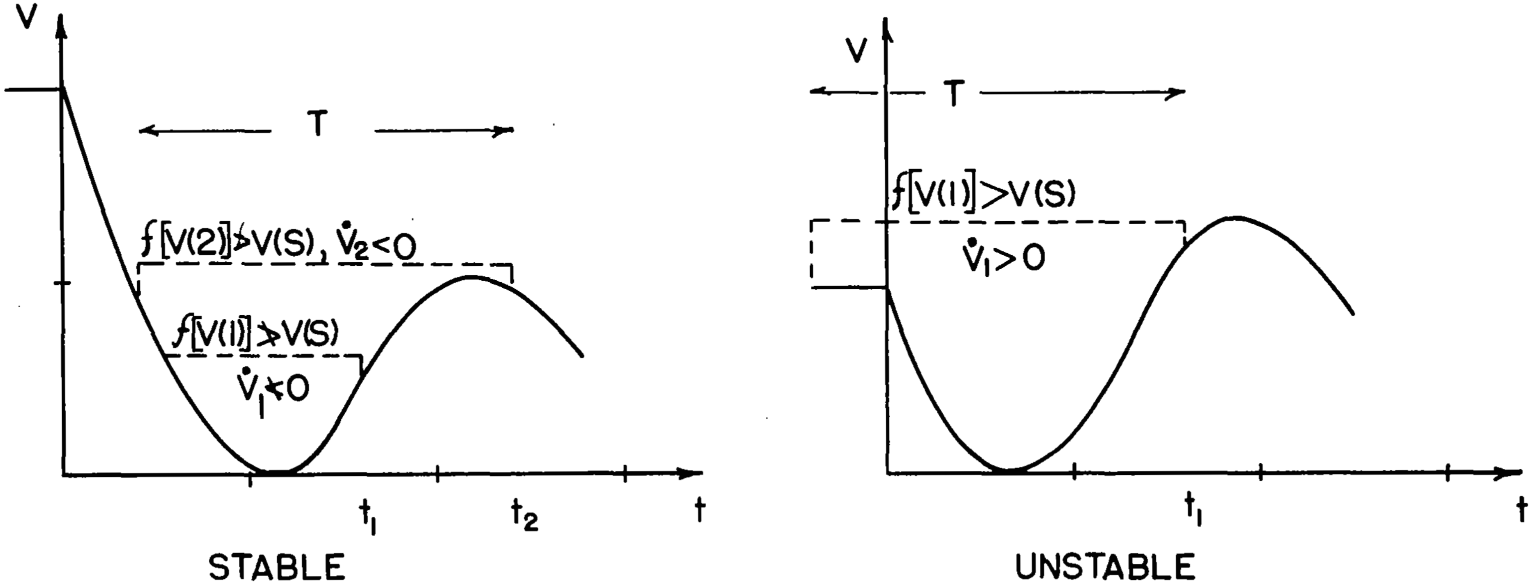


FIGURE 3.5.

STABILITY IN TIME DELAY SYSTEMS

Thus for the range of parameters (3.11), V eventually decreases for all $t - T \leq S \leq t$ and the system is asymptotically stable independent of the initial conditions and the magnitude of the delay. The parameter q is arbitrary subject to $0 < q < 1$; but q very close to unity in (3.11) leads to the greatest range of parameters for which (3.10) is sign-definite.

Razumikhin (1960) suggests a substitution which provides stability results as a function of T. The idea is based upon the use of the mean value of the solution over any $t - T \leq S \leq t$. If $x(\sigma)$ is the mean value of x(s) over the interval, the area under the curve equals the product of the mean value and the width of the interval, or

$$Tx(\sigma) = \int_{t-T}^{t} x(s)\,ds \tag{3.12}$$

The time derivative of Eq. (3.12) gives, since $\dot{x}(\sigma) = \frac{dx}{d\sigma}\frac{d\sigma}{dt}$,

$$x(t-T) = x(t) - T\,\dot{x}(\sigma) \ . \tag{3.13}$$

The technique is to write the original system equations in terms of σ . In this case,

$$\dot{x}(\sigma) = -ax(\sigma)-bx(\sigma-T) \tag{3.14}$$

If Eq. (3.14) is substituted into Eq. (3.13), it is found that

$$x(t-T) = x(t) + aT\,x(\sigma) + bT\,x(\sigma-T) \ . \tag{3.15}$$

Next, Eq. (3.15) is substituted into $\dot{V}$, which yields

$$\frac{1}{2}\dot{V} = -(a+b)x^2(t) - abTx(t)x(\sigma)-b^2Tx(t)x(\sigma-T) \ ,$$

and, as before,

$$\frac{1}{2}\dot{V} \leq -(a+b)x^2(t) + |abTx(t)x(\sigma)| + |b^2Tx(t)x(\sigma-T)| \quad . \qquad (3.16)$$

Now there exist two inequalities of the form (3.9) -- one for the interval 6 - T to t, and the other from $\sigma - T$ to σ . The choice of q such that

$$x^2(\sigma-T) < \sqrt{\frac{1}{q}}\, x^2(t)$$

and

$$x^2(\sigma) < \sqrt{\frac{1}{q}}\, x^2(t)$$

for any given value of T, describes a more restricted class of curves which satisfies the delay inequalities. Ineq. (3.16) now becomes

$$\frac{1}{2}\dot{V} \leq -\left[(a+b) - \frac{|ab|\,T}{\sqrt{q}} - \frac{b^2T}{\sqrt{q}}\right] x^2(t) \ ,$$

which consists of the two separate cases,

$$\frac{1}{2}\dot{V} \leq -\left[a(1-bT)+b(1-bT)\right] x^2(t)$$

for $a > 0$, $b > 0$, and

$$\frac{1}{2}\dot{V} \leq -\left[a(1+bT) + b(1-bT)\right] x^2(t)$$

for $a < 0$, $b > 0$. Note that q is chosen close to unity.

The negative-definiteness of the right sides of these two inequalities is ensured if

$$aT > 0 \quad , \; 0 < bT < 1 \quad ;$$

$$b > 0 \; , \; \frac{-bT(1-bT)}{(1+bT)} < aT < 0 \quad . \tag{3.17}$$

Fig. 3.6 illustrates the region of stable parameters from (3.11), the case independent of the delay. Figure 3.7 shows the result obtained in (3.17) along with the total region obtained by superimposing the two results. This technique is valid due to the sufficiency of the stability conditions provided by the Second Method. The maximum value of $-aT$ in Fig. 3.7 occurs at $b = \sqrt{2} - 1$ and equals $2\sqrt{2} - 3$. The results are seen to be conservative while providing a good estimate of the true region.

The boundary of the exact region shown in Fig. 3.7 and 3.8 is found by taking the Laplace Transform of Eq. (3.7) and solving the resulting characteristic equation. If $p = j\omega$ is the Laplace Transform variable, the equation is

$$p + a + b \exp(-pT) = 0$$

Since

$$\exp\left[-j\omega T\right] = \cos\omega T - j \sin \omega T \quad ,$$

the parametric equations for the roots of the characteristic equation are

$$aT = -\omega T \cot \omega T \; ,$$

$$bT = \left[(\omega T)^2 + (aT)^2\right]^{1/2} \tag{3.18}$$

and for $T = 0$,

$$p = -(a+b) \quad .$$

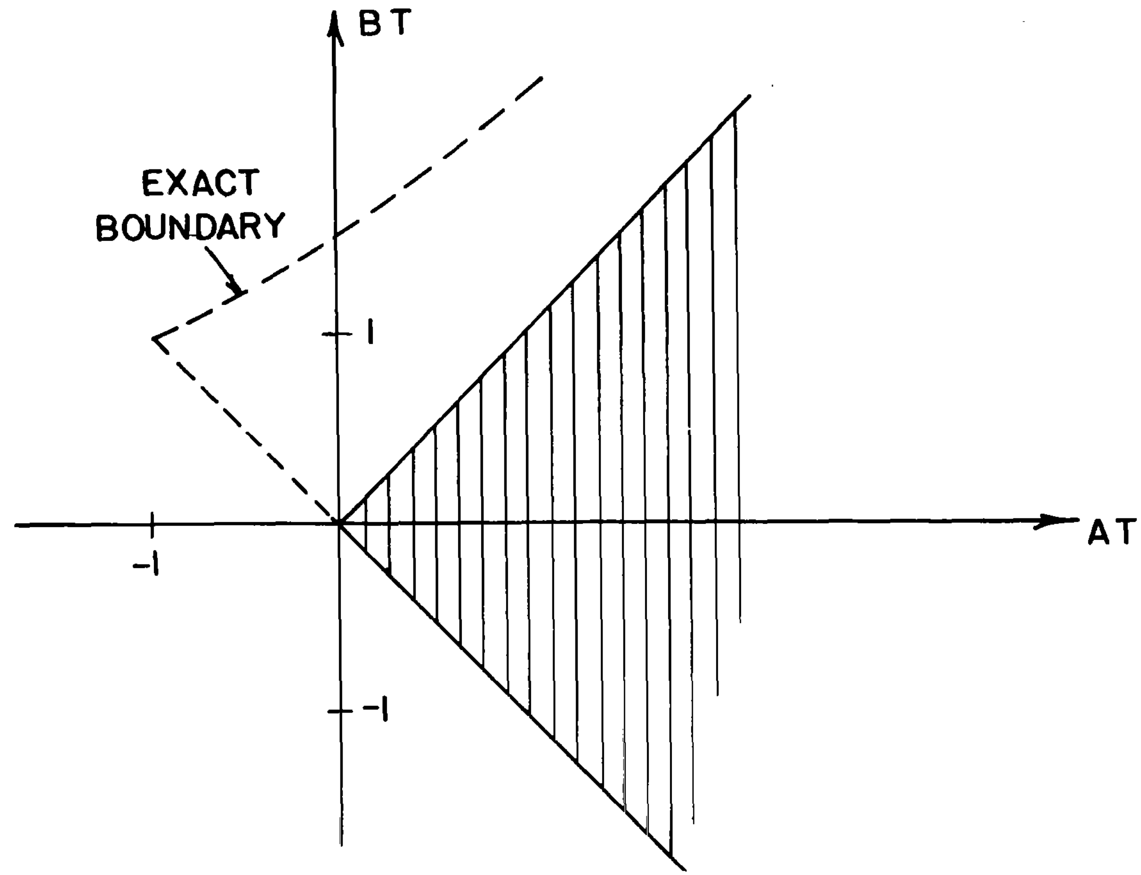
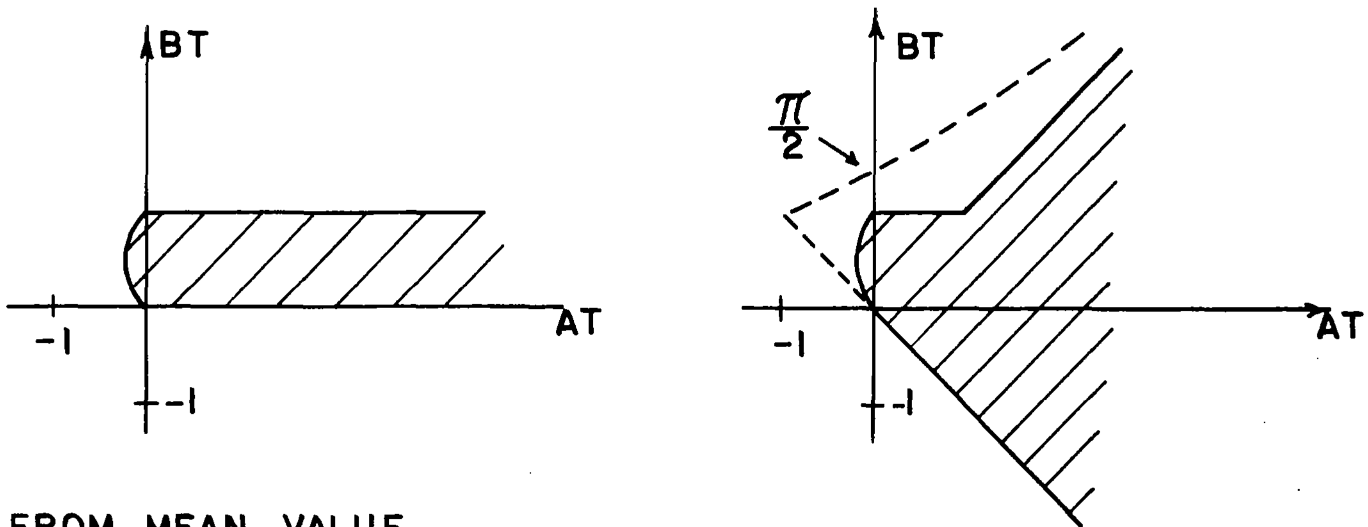


FIGURE 3.6.

STABLE PARAMETERS, $\dot{X} = -AX - BX(t-T)$



FROM MEAN VALUE REPRESENTATION

COMBINED RESULT

FIGURE 3.7.

STABLE PARAMETERS OF $\dot{X} = -AX - BX(t-T)$

In order that all solutions decrease exponentially, all roots of the characteristic equation must have negative real parts. The lower bound is then

$$(a + b) > 0$$

and the upper bound results from Eq. (3.18).

An approximate answer results if the lowest order Pade Approximant (Weaver, 1963, pp. 75) is used. For example, if

$$\exp(-pT) \approx \frac{2-pT}{2-pT} \quad ,$$

the resulting algebraic characteristic equation has roots with negative real parts if

$$aT > - bT$$

$$bT < 2 + aT$$

The exact and approximate regions of stable parameters appear in Fig. 3.8. The approximate approach results in an overestimate of the region of stability. This is a highly undesirable situation if a practical problem is under consideration. The exact result can be found from the characteristic equation only if the system is linear. For this reason and because the process becomes extremely unwieldy for high order systems, the approach is of little interest. The point is that any such approximation should be ruled out on practical grounds. The delay term could in fact be represented as a truncated Taylor series

$$x(t-T) \approx x(t) - T\dot{x}(t)$$

and incorporated into a nonlinear system. Again, there can be no guarantee as to the quality of the result.

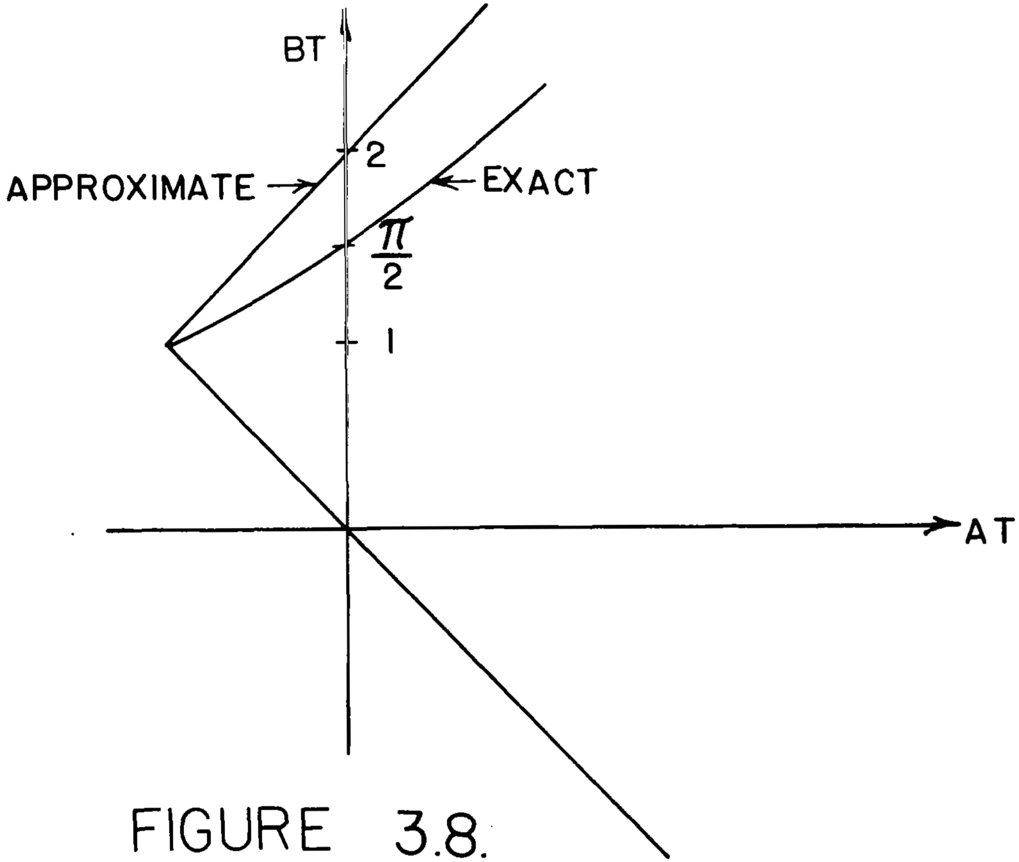


FIGURE 3.8.

SERIES APPROXIMATION

The concept of a Liapunov functional should be considered briefly because it seems at first to be natural to the time delay problem. Krasovskii (1963) suggests the use of the form

$$V = x^2(t) + a \int_{t-T}^{t} x^2(s)ds \ ,$$

for the system (3.7). The time derivative along solutions of (3.7) is

$$\dot{V} = - ax^2(t) - 2bx(t)x(t-T) - ax^2(t-T)$$

which is negative-definite if

$$a > 0 \ ; \ |b| < a$$

This is the same result as found previously using $V = x^2$. This approach is natural because it yields a negative-definite $\dot{V}$ for all time.

The difficulty with the method is twofold. First, x(t) and x(t-T) are treated as two distinct state variables in $\dot{V}$. This increases the dimensionality of the system. Secondly, it appears that for a nonlinear problem, no region of stability can be found due to the unstationary nature of V. An actual solution in time must be found to evaluate V and therefore the region of stability. The method is useful only for low order linear systems.

Higher Order Systems

Ineq. (3.9) is a relatively simple relationship for the first order case. If the system is, for example, second order and if

$$V = x_1{}^2(t) + x_2{}^2\ (t) \quad ,$$

then Ineq. (3.9) is

$$x_1^2(s) + x_2^2(s) < \frac{1}{q}\left[x_1^2(t) + x_2^2(t)\right].$$

Because the delayed variables appear in the equations as $x_1(s)$ or $x_2(s)$ alone, it is necessary that the inequality be rewritten to obtain the variables in an isolated, useful form. In this case it is clear that

$$x_1^2(s) < \frac{1}{q}\left[x_1^2(t) + x_2^2(t)\right]$$

and

$$x_2^2(s) < \frac{1}{q}\left[x_1^2(t) + x_2^2(t)\right]$$

$$|x_1(s)| \text{ or } |x_2(s)| < \frac{1}{\sqrt{q}}\sqrt{x_1^2(t) + x_2^2(t)} \quad .$$

The relationship

$$x_1^2 + x_2^2 < (|x_1| + |x_2|)^2$$

must hold due to the addition of positive cross products on the right side. Therefore

$$|x_1(s)| \text{ and } |x_2(s)|$$

are independently less than

$$\frac{1}{\sqrt{q}}(|x_1(t)| + |x_2(t)|) ,$$

and in general, for an nth-order system,

$$|x_i(s)| < \frac{1}{\sqrt{q}} \sum_{j=1}^{n} |x_j(t)| \tag{3.19}$$

describes a larger class of curves which satisfies the original inequality. This is a less restrictive condition mathematically; but the number of solutions which actually satisfy (3.19) may be so small that the range of validity of the results is severely restricted.

Furthermore, if V is the general quadratic form

$$V = \underline{x}^T \, \underline{B} \, \underline{x} \; ,$$

the class of curves which satisfy the inequalities is enlarged further. For example, if

$$V = x_1{}^2 + x_1 x_2 + x_2{}^2 \quad ,$$

the only way that x_1 or x_2 can be obtained separately such as in Ineq. (3.19), is to restrict the sum of the squares of x(t) to be greater than the sum of the squares of the x(s). The sum of the squares of either variable is the square of the radius of an n-dimensional sphere, thus it is necessary to find the ratio of the largest to the smallest sphere which uniquely intersects the n-dimensional elliptical form.

The process is shown geometrically in Fig. 3.9 for the second order case under consideration. The ratio of the radii-squared is 3 , therefore the inequality relationship is

$$|x_1(s)| \text{ or } |x_2(s)| < \sqrt{\frac{3}{q}} \left[\; |x_1(t)| + |x_2(t)| \; \right] \; .$$

Clearly, the additional factor $\sqrt{3}$ will cause the method to become restrictive if the delay inequality substitutions result in positive quadratic terms in $\dot{V}$. This is, in fact, the case. In the general nth-order case, the spherical intersections of the quadratic space satisfy

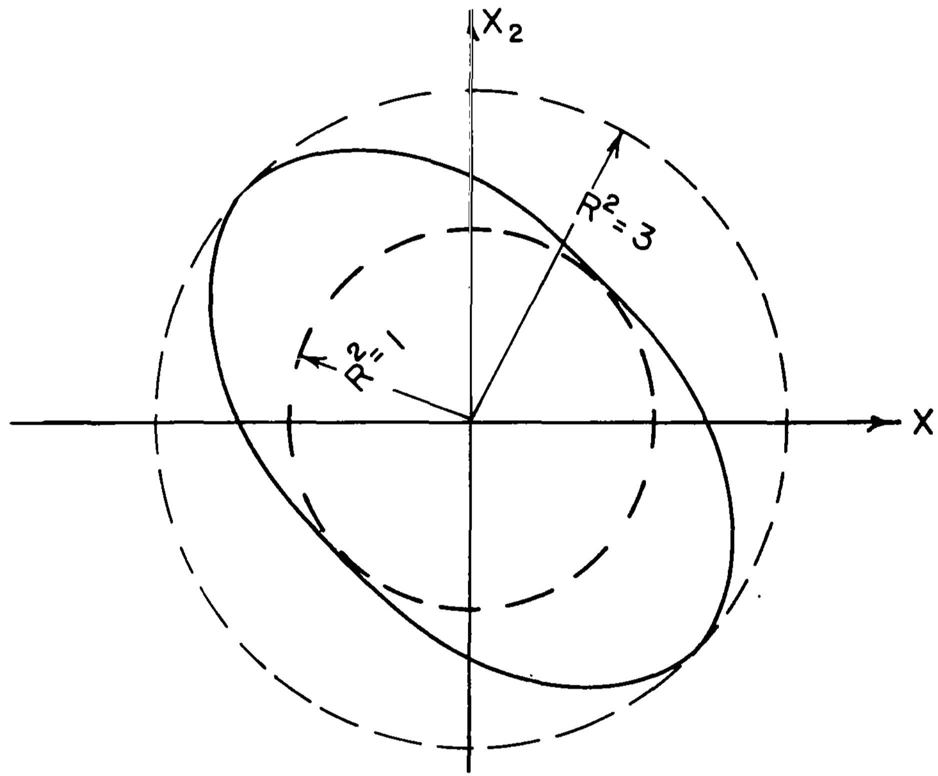


FIGURE 3.9.

CIRCLE-ELLIPSE RELATIONSHIPS

the spherical intersections of the quadratic space satisfy

$$\underline{x}^T \underline{B} \, \underline{x} = \underline{x}^T r^2 \underline{I} \, \underline{x} \quad ,$$

or

$$| \underline{B} - r^2 I | = 0 \, , \tag{3.20}$$

and the inequalities are

$$|x_i(s)| < \frac{1}{\sqrt{q}} \sqrt{\frac{r^2_{max}}{r^2_{min}}} \sum_{j=1}^{n} |x_j(t)| \tag{3.21}$$

where r_{max} and r_{min} are the maximum and minimum solutions to the determinant (3.20). Again for the case illustrated geometrically,

$$\underline{B} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

and

$$|\underline{B} - r^2 \underline{I}| = \begin{bmatrix} 1-r^2 & \frac{1}{2} \\ \frac{1}{2} & 1-r^2 \end{bmatrix} = 0 \quad ,$$

or

$$r^2 = 1, \; 3 \; .$$

It is stressed that while Ineq. (3.21) constitutes an approximation, the approximation is made within the framework of the Second Method. Therefore the sufficiency of the stability conditions are always ensured.

This development also demonstrates the advantage of a quadratic V. If the $\underline{B}$ matrix is a function of x , the radii relationships also depend upon x. The meaning of this is that the actual shape of V changes with x , so there is no consistent geometrical relationship between the unique spheres and the extreme points of V . If the solutions to Eq. (3.20) are negative or complex, there are no satisfactory relationships. However, if the V functions represent real closed surfaces, there should be real solutions to at least the third order case.

Chapter 4

COUPLED-CORE STABILITY

It is necessary first to study the stability of a simplified coupled-core model. The method set forth in the preceding chapter for time delay systems relies upon a quadratic Liapunov function. This type of V function works quite well for the first-order linear example. However, it is pointed out again that by necessity the method is restrictive for higher order problems. The effect of system nonlinearities upon the method, moreover, remains to be discovered.

A Simplified Model

Even the simplest coupled-core model results in a difficult stability problem. For a system of two identical cores, if delayed neutrons are neglected and if the intrinsic negative reactivity effect is proportional to the power, Eq. (13a) describes the system as follows:

$$\dot{x}_1(t) = \frac{-C_P P_{10}}{\ell} x_1(t)\left[1+x_1(t)\right] - \frac{\alpha}{\ell}\frac{P_{20}}{P_{10}} x_1(t) + \frac{\alpha}{\ell}\frac{P_{20}}{P_{10}} x_2(t-T) \ ,$$

$$\dot{x}_2(t) = \frac{-C_P P_{20}}{\ell} x_2(t)\left[1+x_2(t)\right] - \frac{\alpha}{\ell}\frac{P_{10}}{P_{20}} x_2(t) + \frac{\alpha}{\ell}\frac{P_{10}}{P_{20}} x_1(t-T) \ .$$

The coupling coefficients and delay times are equal between the cores.

Although the delayed neutrons are neglected, it is convenient from a notational standpoint to describe the intrinsic reactivity and coupling contributions to the equations in terms of $\frac{\beta}{\ell}$, a measurable and important parameter in any physical reactor system. If the two cores are in a design operating state with identical power outputs,

$$\frac{C_P P_{10}}{\ell} = \frac{C_P P_{20}}{\ell} = k_P \frac{\beta}{\ell}$$

and

$$\frac{\alpha}{\ell} \quad \frac{P_{20}}{P_{10}} = \frac{\alpha}{\ell} = k_c \frac{\beta}{\ell} \quad ,$$

where k_P and k_c are the fractions of β/ℓ contributed by the internal reactivity and coupling, respectively. The kinetics equations become

$$\dot{x}_1(t) = - k_p \frac{\beta}{\ell} x_1(t) \left[1+x_1(t)\right] - k_c \frac{\beta}{\ell} x_1(t) + k_c \frac{\beta}{\ell} x_2(t-T)$$

$$\dot{x}_2(t) = - k_p \frac{\beta}{\ell} x_2(t) \left[1+x_2(t)\right] - k_c \frac{\beta}{\ell} x_2(t) + k_c \frac{\beta}{\ell} x_1(t-T) \tag{4.1}$$

where x_1 and x_2 are the fractional changes of the power from equilibrium in the two cores.

Stability Without Time Delay

No reference is made to the time delay in choosing a quadratic V function. This implies that the stability investigation necessarily begins with an analysis of the system with the delay neglected. Additionally, the approach to the problem via the quadratic V function requires that the linear system is stable.

Equation (4.1) with $T = 0$, are seen to be representable in the general matrix form (3.5) since the nonlinearities are separable. The linearized equations are

$$\dot{x}_1 = -\frac{\beta}{\ell}(k_p+k_c)x_1 + \frac{\beta}{\ell}k_c x_2 \qquad (4.2)$$

$$\dot{x}_2 = -\frac{\beta}{\ell}(k_p+k_c)x_2 + \frac{\beta}{\ell}k_c x_1$$

The argument is deleted from the state variables for convenience, because $T = 0$. The nonlinear portion of the i-th equation is

$$-k_p \frac{\beta}{\ell} x_i^2 \quad . \qquad (4.3)$$

It is instructive to make use of the canonic system representation for determining a V function. In the second-order case, this is a relatively simple procedure, providing a quick determination of linear stability through the eigenvalues. The linear system $\underline{A}$ matrix is

$$\underline{A} = \frac{\beta}{\ell}\begin{bmatrix} -(k_p+k_c) & k_c \\ k_c & -(k_p+k_c) \end{bmatrix} ,$$

and accordingly the eigenvalues satisfy

$$\psi^2 + 2\frac{\beta}{\ell}(k_p+k_c)\,\psi + \left(\frac{\beta}{\ell}\right)^2 k_p(k_p + 2k_c) = 0 \ .$$

The solutions to the eigenvalue equation cannot be positive or zero as long as $k_p > 0$. Therefore the linearized system is asymptotically stable if there exists a non-zero negative intrinsic reactivity effect. The coupling effect k_c is restricted physically to positive values.

The canonic V is

$$V = Z_1^2 + Z_2^2$$

where, from the transformation (3.6),

$$\underline{Z} = \underline{P}^{-1}\,\underline{x}$$

The elements of the $\underline{P}$ matrix are found by solving

$$\underline{A}\,\underline{P} = \underline{P}\,\underline{\Lambda}$$

For this case,

$$\underline{P} = \begin{bmatrix} K_1 & K_2 \\ K_2 & -K_1 \end{bmatrix}$$

and

$$\underline{P}^{-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{K_1} & \frac{1}{K_1} \\ \frac{1}{K_2} & -\frac{1}{K_2} \end{bmatrix},$$

where K_1 and K_2 are arbitrary positive constants. The V function in the real system variables is therefore

$$V = x_1^2 + 2K\,x_1x_2 + x_2^2 \qquad (4.4)$$

where

$$K = \frac{K_1^2 - K_2^2}{K_1^2 + K_2^2}\;.$$

By Theorem 3.2, K is restricted to values $|K| < 1$.

This approach provides some hints for choosing general quadratic forms. The linear system matrix $\underline{A}$ in this case is symmetric. Also, the resulting V is seen to be symmetric in x_1 and x_2 . This idea would eliminate some guesswork in a non-methodical approach. The concept can be generalized further. The canonic approach results in a V function in which the variables are weighted according to the magnitudes of their coefficients in the system equations. This provides a general guide for deciding upon the coefficients in a general V . There is no doubt that these results could be discovered through the process of constraining $\dot{V}$ to be negative definite, but only after much trial and error.

The derivative of Eq. (4.4) is

$$\frac{1}{2}\dot{V} = \dot{x}_1(x_1 + Kx_2) + \dot{x}_2(x_2 + Kx_1) \qquad (4.5)$$

which is, along solutions of the linearized system (4.2)

$$\frac{1}{2}\frac{\ell}{\beta}\dot{V} = -\left[(k_p+k_c)-Kk_c\right](x_1^2+x_2^2)$$

$$-2x_1x_2\left[K(k_p+k_c)-k_c\right] . \qquad (4.6)$$

Equation (4.6) is negative-definite if

$$k_p + (1-K)k_c > 0 ,$$

$$k_p(k_p + 2k_c) > 0 .$$

These inequalities are satisfied for all admissible values of K if k_p and k_c are positive. This result, of course, is predetermined in the canonic approach by the sign of the eigenvalues.

In order to illustrate clearly the process of obtaining the region of stability for the nonlinear system, it is assumed that $k_p = k_c = 1$. Equation (4.5) for the nonlinear system (4.1) is

$$\frac{1}{2}\frac{\ell}{\beta}\dot{V} = -(2-K)(x_1^2+x_2^2)-2(2K-1)x_1x_2$$

$$-(x_1^3+x_2^3)-Kx_1x_2(x_1+x_2) .$$

The technique is to choose a value of K and find the region $\dot{V} < 0$. The largest V which fits in this region is the region of stability. A new value of K is then chosen and the process is repeated. If the new region encloses an area in the state space not previously described, the two regions are superimposed to improve the result. If $K = 0$,

$$V = x_1^2 + x_2^2$$

which describes circles in the x_1x_2 plane, and

$$\frac{1}{2}\frac{\ell}{\beta}\dot{V} = -2(x_1^2+x_2^2) + 2x_1x_2 - (x_1^3+x_2^3) \ . \qquad (4.7)$$

The zero solution of Eq. (4.7) can be found easily by transforming the coordinates into polar form

$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta \ .$$

The equation $\dot{V} = 0$ is satisfied at the origin, which must be true by the conditions of the Second Method, and uniquely along the curve described by

$$r = -2\ \frac{1 - \cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta} \ .$$

This describes a straight line $x_2 = -x_1 - 2$, which has a slope of -1 and passes through the point $x_1 = x_2 = -1$.

A few numerical substitutions in Eq. (4.7) reveal that $\dot{V}$ is negative above the line $\dot{V} = 0$, or in the direction of the origin. The largest circle which fits in this region passes through the point $x_1 = x_2 = -1$; therefore the nonlinear system with zero delay is asymptotically stable in the region

$$x_1^2 + x_2^2 < 2 \ , \qquad (4.8)$$

or within a circle of radius $\sqrt{2}$ in the x_1x_2 state plane.

For all values of $|K| < 1$, the line $\dot{V} = 0$ passes through the -1,-1 point. Values of $K < 0$ result in ellipses whose major axes fall along $x_1 = x_2$. Hence, regions obtained for all $K < 0$ lie inside the region (4.8) and add no new information. Values of $K > 0$ describe ellipses elongated on $x_1 = -x_2$ and the curves $\dot{V} = 0$ in polar form have minimum radii other than $\sqrt{2}$. The calculations are repeated for $K = 1/2$ and $K = 3/4$ and the regions are superimposed to obtain a larger stable region.

Figure 4.1 shows the result of these calculations along with some actual stable and unstable system trajectories obtained from analog computer solutions. The region is symmetric about $x_1 = x_2$. The line $x = -1$ is the line of zero power for the system. Thus, trajectories outside the region $x > -1$ do not represent physically real cases. For this practical reason, the region obtained for $K = 3/4$ is superfluous.

Stability with Time Delay

The basic approach for treating the nonlinear time delay problem also is first to constrain the linearized system to be asymptotically stable. The linearized system with delay is, from Eq. (4.1), for the case $k_p = k_c = 1$,

$$\begin{aligned} \dot{x}_1 &= -2\frac{\beta}{\ell}x_1 + \frac{\beta}{\ell}x_2(t-T) \\ \dot{x}_2 &= -2\frac{\beta}{\ell}x_2 + \frac{\beta}{\ell}x_1(t-T) \ , \end{aligned} \tag{4.9}$$

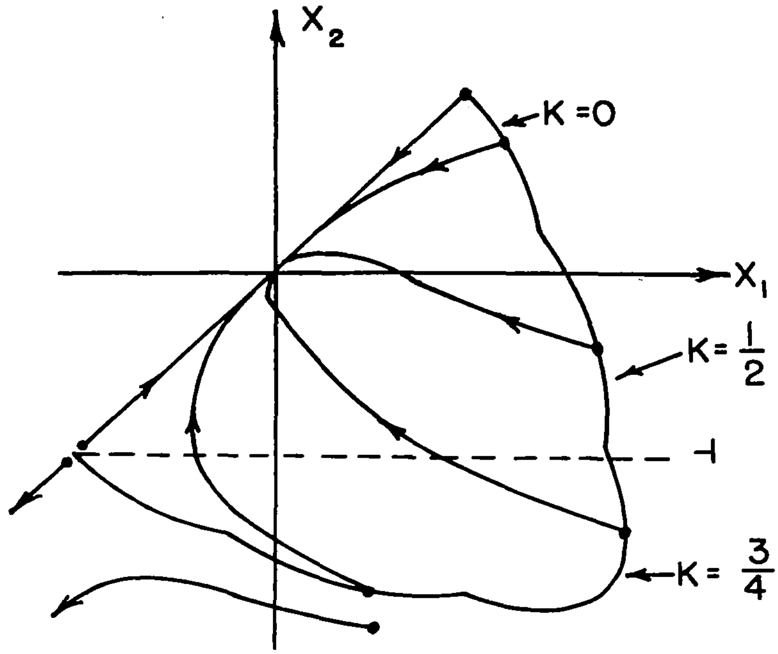


FIGURE 4.1.

REGION OF STABILITY
WITHOUT DELAY

where the argument of x is deleted if it is only t.

If K = 0, from Eq. (4.5), $\dot{V}$ is

$$\frac{1}{2}\frac{\ell}{\beta}\ \dot{V} = -(x_1{}^2+x_2{}^2) + \frac{1}{2}\left[x_1x_2(t-T)+ x_2x_1(t-T) \right] . \tag{4.10}$$

From the mean value representation in Eq. (3.13) and from the system equations (4.9), the delayed variable substitutions are

$$\begin{aligned} x_1(t-T) &= x_1 + 2\,\frac{\beta}{\ell}\,Tx_1(\sigma) - \frac{\beta}{\ell}\,Tx_2(\sigma-T) , \\ x_2(t-T) &= x_2 + 2\,\frac{\beta}{\ell}\,Tx_2(\sigma) - \frac{\beta}{\ell}\,Tx_1(\sigma-T) . \end{aligned} \tag{4.11}$$

Equations (4.11) are now substituted into Eq. (4.10) and, according to the technique described in the previous chapter, the following steps are taken:

(a) The cross products $xx(\sigma)$, $xx(\sigma-T)$ are written as $|xx(\sigma)|$, and $|xx(\sigma-T)|$;

(b) Wherever the absolute value cross products appear, the sign of the cross product coefficient is made positive to ensure that the resulting function is greater than $\dot{V}$.

The result is

$$\frac{1}{2}\frac{\ell}{\beta}\,\dot{V} \leq -(x_1{}^2+x_2{}^2)+ x_1x_2 \tag{4.12}$$

$$+ \frac{\beta}{\ell}\,T \left[|x_1x_2(\sigma)|+|x_2x_1(\sigma)|\right]+ \frac{1}{2}\left[|x_1x_1(\sigma-T)|+|x_2x_2(\sigma-T)|\right] .$$

The inequality relationships (3.9) arising from Theorem 3.4 are given by Ineq. (3.19) for the case $K = 0$. Therefore the magnitudes

$$|x_1(\sigma)|, \quad |x_2(\sigma)|, |x_1(\sigma-T)|, \quad |x_2(\sigma-T)|$$

must all be individually less than

$$\frac{1}{\sqrt{q}} \left[|x_1| + |x_2| \right] \quad .$$

If the inequality relationships are substituted into Ineq. (4.12), the result is

$$\frac{1}{2}\frac{\ell}{\beta}\dot{V} \leq -(1 - \frac{3}{2}\frac{\beta}{\ell} T)(x_1^2 + x_2^2) + (1 + 3\frac{\beta}{\ell} T)|x_1 x_2| \quad . \tag{4.13}$$

The right side of Ineq. (4.13) is $-w(x)$ from Theorem 3.4, and $-w(x)$ is negative-definite if

$$\frac{\beta}{\ell} T < \frac{1}{6} \quad .$$

The extension of the manipulations to the nonlinear system is complicated by the fact that the nonlinear terms appear not only with the argument t but also with the argument σ . This of course is due to the mean value representation wherein the complete system equations must be rewritten in terms of $\dot{x}(\sigma)$. The final result is

$$\frac{1}{2}\frac{\ell}{\beta}\dot{V} \leq -(1-\frac{3}{2}\frac{\beta}{\ell} T)(x_1^2+x_2^2)+(1+\frac{3}{2}\frac{\beta}{\ell} T)|x_1 x_2|$$
$$- \frac{1}{2}(1-\frac{\beta}{\ell} T)(x_1^3+x_2^3) + \frac{1}{2}\frac{\beta}{\ell} T x_1 x_2 (x_1 + x_2) . \tag{4.14}$$

The case of $K = 1/2$ yields

$$V = x_1^2 + x_1 x_2 + x_2^2$$

which leads to the elliptical delay inequality relationships (3.21). The solution to Eq. (3.20) for $K = 1/2$ is

$$\frac{r_{max}^2}{r_{min}^2} = 3 \quad ,$$

therefore all the absolute values of the variables with retarded arguments must be less than

$$\sqrt{\frac{3}{q}} \left[|x_1| + |x_2| \right] \quad .$$

The result for the linearized system is

$$\frac{2}{3} \frac{\ell}{\beta} \dot{V} \leq -(1-3\sqrt{3} \frac{\beta}{\ell} T)(x_1^2+x_2^2)+ 6\sqrt{3} \frac{\beta}{\ell} T \ |x_1 x_2| \tag{4.15}$$

and for the nonlinear case,

$$\frac{2}{3} \frac{\ell}{\beta} \dot{V} \leq -(1 - 3\sqrt{3} \frac{\beta}{\ell} T)(x_1^2+x_2^2)+ 6\sqrt{3} \frac{\beta}{\ell} T \ |x_1 x_2| \tag{4.16}$$
$$-(\frac{2}{3} - 3 \frac{\beta}{\ell} T)(x_1^3+x_2^3)-(\frac{1}{3} - 3 \frac{\beta}{\ell} T)x_1 x_2 (x_1+x_2) \quad .$$

The right side of Ineq. (4.15) is negative-definite if

$$\frac{\beta}{\ell} T < \frac{\sqrt{3}}{18} \quad .$$

The regions of stability obtained are shown in Fig. 4.2 for different values of $\frac{\beta}{\ell} T$. It is noted that for the case $K = 1/2$, $\frac{\beta}{\ell} T = 0.1$, there is no region added to the circular region. This is due to the fact that linear asymptotic stability cannot be found in this case. $\frac{\beta}{\ell} T$ must be less than $\frac{\sqrt{3}}{18} = 0.096$, which is clearly not true. This emphasizes the added restrictiveness of the elliptical V function for the time delay problem.

The actual effect of the delay on some typical trajectories is shown in Fig. 4.3 for the example under consideration. The time delay analog simulation is presented in Appendix A. It is seen that the upper two solutions exhibit the characteristics of a system that is becoming less stable, since the path lengths are generally longer. The lower trajectory, however, actually becomes stable in the presence of the delay.

A Complete Parametric Stability Study

The techniques can be applied to the system for a general k_p, k_c, and T . This can be done readily if only one V function is used. The circular form is especially appropriate because the resulting region lies almost entirely within the physically real part of the state plane. Again, the circular function is also the least restrictive for the time delay problem. Probably the most important advantage however is that of ease of visualization and expression of the results. For example, even if the example is tenth order, the stability result is described simply as

$$| \underline{x} | < R$$

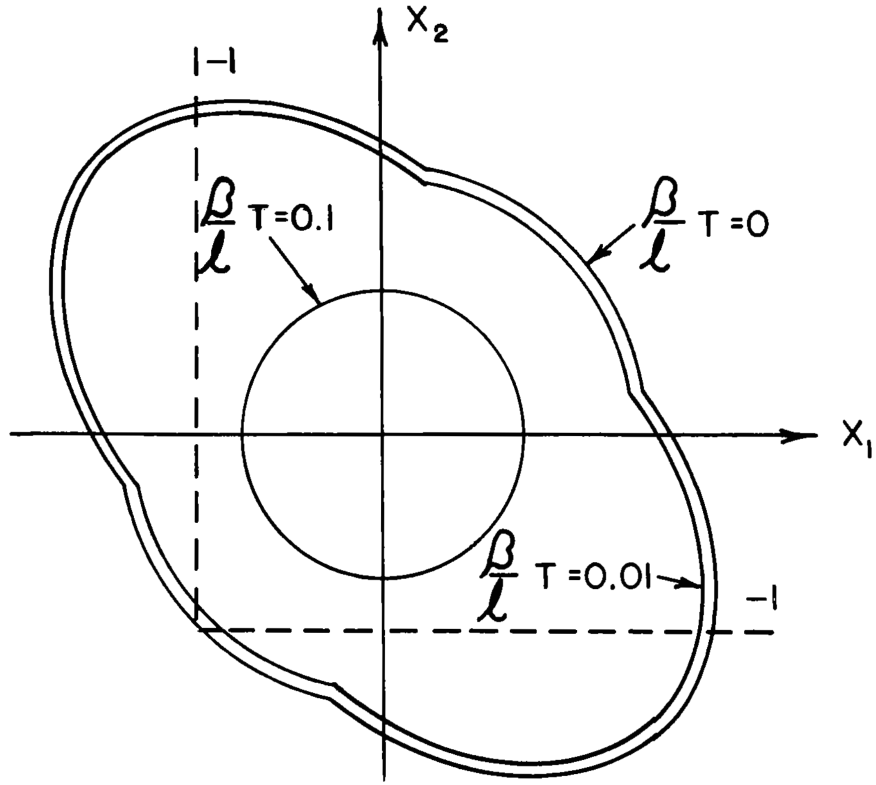


FIGURE 4.2.

REGIONS OF STABILITY WITH DELAY

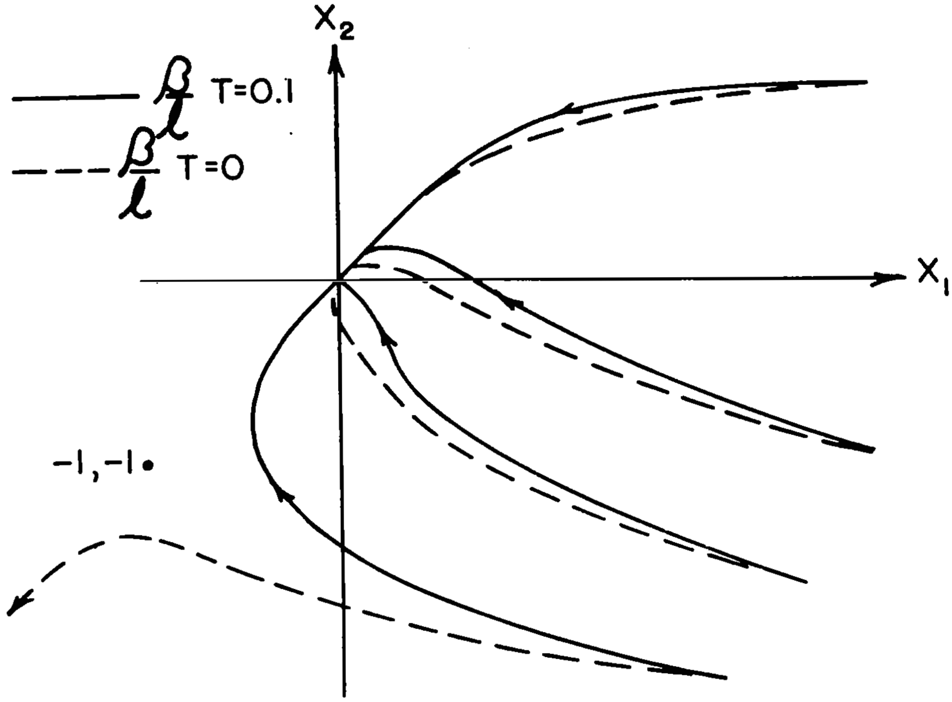


FIGURE 4.3.

EFFECT OF DELAY ON TRAJECTORIES

where $| \underline{x} |$ is the length of the vector whose elements are the magnitudes of the initial conditions of all the state variables. R is the radius of a sphere of the same dimensionality as the system. R is also the radius of the largest V in the region $\dot{V} < 0$.

One case yet to be considered is that in which the result is independent of delay. Here, the mean value substitution is not used and the inequality relationships

$$|x_1(t-T)|, \quad |x_2(t-T)| < \frac{1}{\sqrt{q}} \left[|x_1|+|x_2|\right]$$

are used directly in the appropriate $\dot{V}$. The result is, for the linearized case, from Eq. (4.10),

$$\frac{1}{2} \frac{\ell}{\beta} \dot{V} \leq - k_p(x_1{}^2+ x_2{}^2)+ 2k_c |x_1x_2|$$

and asymptotic stability is concluded if

$$\frac{k_c}{k_p} < 1 .$$

The nonlinear result is

$$\frac{1}{2} \frac{\ell}{\beta} \dot{V} \leq -k_p(x_1{}^2+x_2{}^2)-k_p(x_1{}^3+x_2{}^3)+2k_c |x_1x_2| .$$

The complete nonlinear stability results are presented in Fig. 4.4 in terms of the radius of the region of stability for $V = x_1^2 + x_2^2$. Probably the most interesting conclusion that can be made from Fig. 4.4 is that as long as k_c is less than k_p , there exists a finite region of stability independent of T . From the numbers given at the end of Chapter 2, some practical system parameters are

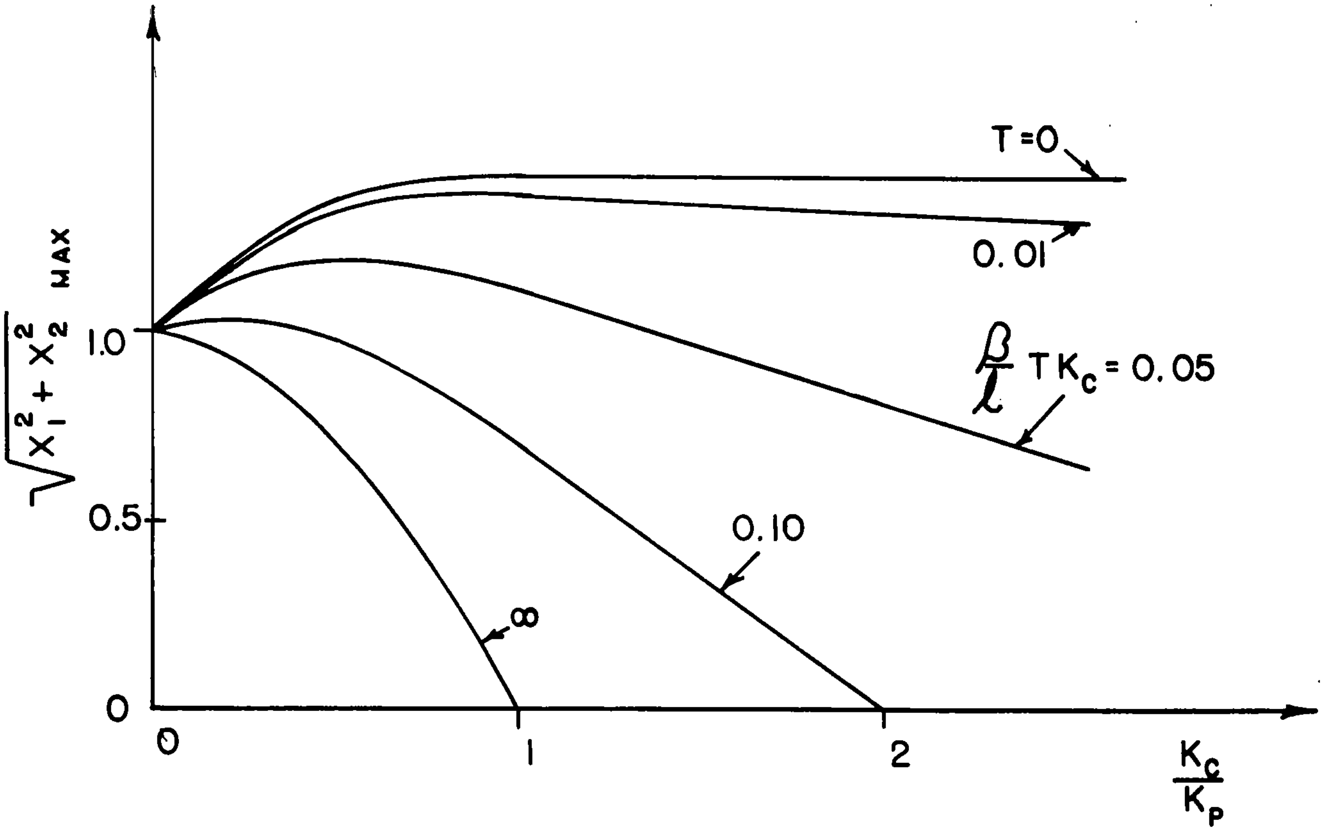


FIGURE 4.4.

NONLINEAR STABILITY, TWO CORE SYSTEM

$$\frac{k_c}{k_p} = 0.1$$

and

$$\frac{\beta}{\ell} T\, k_c = 0.01 \quad .$$

This point on the curve practically coincides with the case $T = 0$. The radius of the region of stability is slightly greater than unity. This means that the power level of either reactor may be perturbed up to 100% and the system still returns to its operating point. At 2000 megawatts this number is extremely generous.

The general conclusion is that as the magnitude of the coupling increases with respect to the negative reactivity effect, the effect of the delay time on stability becomes more apparent. This is quite reasonable since the coupling term in the equations is the one with the retarded argument. For ratios $\frac{k_c}{k_p} > 1$, very small increases in T are seen to decrease greatly the size of the known region of stability. This effect clearly demonstrates that the intrinsic reactivity phenomenon has a strong stabilizing influence on the nonlinear system.

Chapter 5

REALISTIC COUPLED-CORE SYSTEMS

The simplified model used to illustrate the basic method in the previous chapter is not expected to yield a particularly accurate description of the behavior of the real system. The assumption inherent in the simplified model is that the temperature in the core increases directly with the power level. That this is not true can be seen from the system equations, Eq. (2.13). If there are no delayed neutrons, these equations are

$$\begin{aligned}
\dot{x}_1 &= -\frac{\beta}{\ell} k_t z_1(1+x_1) - \frac{\beta}{\ell} k_c x_1 + \frac{\beta}{\ell} k_c x_1(t-T) \\
\dot{z}_1 &= \omega x_1 - \omega z_1 \\
\dot{x}_2 &= -\frac{\beta}{\ell} k_t z_2(1+x_2) - \frac{\beta}{\ell} k_c x_2 + \frac{\beta}{\ell} k_c x_1(t-T) \\
\dot{z}_2 &= \omega x_2 - \omega z_2
\end{aligned} \tag{5.1}$$

where k_t is the temperature reactivity contribution in terms of $\frac{\beta}{\ell}$ at the operating point.

An examination of Eqs. (5.1) reveals that if the power variable x is a step function, that is, if x is zero and suddenly takes on some positive constant values, z has a solution in time of

$$z = x(1 - e^{-\omega t}) \quad .$$

Obviously, for a given change in x , it takes a considerably long period of time for z to achieve the final value of x. The lag between power and temperature is not zero in any event.

The intrinsic reactivity effect, it was concluded previously, has a strong stabilizing effect upon the system. The lag between the power and temperature, not to be confused with a discrete time delay such as in the neutronic coupling effect, undoubtedly leads to a less stable situation. This case also, because it is a higher order problem, leads to increasingly severe stability conditions within the method for treating the time delay. It is necessary, therefore, to apply the method to a realistic problem in order to see if stability can be determined despite the mounting difficulties. It is clearly desirable to discuss practical stability and thus to illustrate the usefulness of a method which appears to be largely mathematical in nature.

The Approach for Higher Order Systems

By this time it is apparent that a judicious choice for a Liapunov function is that of the spherical type wherein only the system variables squared appear. This is necessary to relieve as much as possible the restrictiveness of the method for higher order time delay cases. If the delayed neutrons are neglected for now, such a V function is

$$V = x_1^2 + x_2^2 + Kz_1^2 + Kz_2^2$$

where K is an arbitrary positive constant. Along solutions of the linearized system with no delay, $\dot{V}$ is

$$\frac{1}{2}\dot{V} = -\frac{\beta}{\ell} k_c (x_1^2+x_2^2) + 2\frac{\beta}{\ell} k_c x_1 x_2 - K\omega(z_1^2+z_2^2)$$

$$- (\frac{\beta}{\ell} k_t - K\omega)(x_1 z_1 + x_2 z_2) .$$

Sylvester's determinant for Eq. (5.2) is

$$- \begin{vmatrix} \frac{\beta}{\ell} k_c & \frac{1}{2}(\frac{\beta}{\ell} k_t - K\omega) & -\frac{\beta}{\ell} k_c & 0 \\ \frac{1}{2}(\frac{\beta}{\ell} k_t - K\omega) & K\omega & 0 & 0 \\ -\frac{\beta}{\ell} k_c & 0 & \frac{\beta}{\ell} k_c & \frac{1}{2}(\frac{\beta}{\ell} k_t - K\omega) \\ 0 & 0 & \frac{1}{2}(\frac{\beta}{\ell} k_t - K\omega) & K\omega \end{vmatrix}$$

which leads to a conclusion of negative-definiteness of Eq. (5.2) if the following inequalities are satisfied:

(a) $\frac{\beta}{\ell} k_c > 0$,

(b) $K\omega \frac{\beta}{\ell} k_c - \frac{1}{4}(\frac{\beta}{\ell} k_t - K\omega)^2 > 0$,

(c) $-\frac{1}{4}\frac{\beta}{\ell} k_c (\frac{\beta}{\ell} k_t - K\omega)^2 > 0$,

(d) $-\frac{1}{2}(\frac{\beta}{\ell} k_t - K\omega)^2 \left[k\omega \frac{\beta}{\ell} k_c - \frac{1}{8}(\frac{\beta}{\ell} k_t - K\omega)^2\right] > 0$

A problem arises here. The third inequality can never be satisfied. However, if K is chosen such that

$$K\omega = \frac{\beta}{\ell} k_t \quad ,$$

then the left sides of the last two inequalities are zero. This means that $\dot{V}$ is only negative-semidefinite, or there are combinations of x and z which cause $\dot{V}$ to be zero other than at the origin. If $K\omega = \frac{\beta}{\ell} k_t$, $\dot{V}$ is

$$\frac{1}{2} \dot{V} = - \frac{\beta}{\ell} k_c \; (x_1^2 + x_2^2) + 2 \frac{\beta}{\ell} k_c x_1 x_2 - \frac{\beta}{\ell} k \; (z_1^2 + z_2^2) \; ,$$

and it is seen quickly that the particular solutions for which $\dot{V}$ is zero are

$$x_1 = x_2 \; ,$$
$$z_1 = z_2 = 0 \; .$$

This is actually an admissible situation. From the basic stability theorems, Theorems 3.1 and 3.4, the condition is that $\dot{V}$ must be negative along solutions of the system. Thus if the conditions under which $\dot{V} = 0$ here do not describe system trajectories, asymptotic stability for the linearized system with no delay may still be concluded.

If the values above are substituted into Eq. (5.1), the conclusion is that

$$\dot{x}_1 = 0$$
$$\dot{z}_1 = \omega x_1 \; .$$

The equations for x insist that x must be a constant in time. On the other hand, the equations for z require that z must be changing in time. If z changes, x must also change according to the original equations, Eq. (5.1). The requirements are therefore inconsistent and the solution $x_1 = x_2$, $z_1 = z_2 = 0$ cannot be a solution of the system. This is not true, of course, in the case where the internal reactivity of the system is proportional to the power. In Eq. (4.1) it can be seen that if $x_1 = x_2$ the time rates of change of x_1 and x_2 are the same, specifically

$$\dot{x}_1 = -\frac{\beta}{\ell} k_p x_1(1+x_1) \ ,$$

and as a result $x_1 = x_2$ is a solution of the equations.

The problem of visualizing the results is ever present in higher order problems. This difficulty is partially alleviated by the choice of the spherical V function. In this case, the final results can be expressed in a compact mathematical notation as simply the radius of the n-dimensional sphere. However, the task of finding the region in which $\dot{V}$ is negative remains. The best approach would seem to be to find this region in each plane of the space. This can be accomplished by allowing all the state variables in the function to be zero except for the two which describe the particular plane under consideration. The danger in this approach is that the actual region cannot be found in the entire space merely from the information provided in each plane at the origin. This can be seen in the third order case. If the region is found to be a circle in the xy, xz, and yz planes, it would seem at first that the region could be described as a sphere with the same radius as the three circles. However, a detailed examination might reveal that the surface of the region is actually depressed toward the origin in each quadrant of the three dimensional space.

Nevertheless, there seems to be no other satisfactory approach. Once the suspected region is deduced from the regions in the planes, the technique would be to substitute some values into $\dot{V}$ to ensure that the correct region has been found. A generalized Sylvester's theorem might be employed with the elements of the $\underline{B}$ matrix containing the state variables as well as constants. The success of this approach depends upon whether or not the nonlinearities appear in all the terms. If this is true, the technique becomes prohibitively complex. It is seen from Eqs. (2.13) however, that only the equation which describes the power is nonlinear. The auxiliary equations are linear. In this case, the use of the Sylvester relationships is helpful. In solving the higher order problems, use of both of these approaches will be appropriate; and especially, once a particular problem is solved, intuition will be used to extend the results to additional problems. This is one of the advantages of the Second Method of Liapunov, that previous experience may be drawn upon to improve or to extend the results.

Failure of the Method for High Order Problems

Even though linear asymptotic stability is shown for the system (5.1) with no delay, no stability conclusions result for the case with delay. If the Second Method is applied to the linearized system with delay, $\dot{V}$ is

$$\begin{aligned} \frac{1}{2}\frac{\ell}{\beta}\dot{V} = &- k_c({x_1}^2+{x_2}^2) - k_t({z_1}^2+{z_2}^2) \\ &+ k_c\left[x_1x_2(t-T)+ x_2x_1(t-T)\right] \quad . \end{aligned} \tag{5.2}$$

From the mean value system representation, the substitution for $x_1(t-T)$ is,

$$x_1(t-T) = x_1 + \frac{\beta}{\ell} T \left[k_t z_1(\sigma) + k_c x_1(\sigma) - k_c x_2(\sigma - T)\right] \quad ,$$

and the substitution for $x_2(t-T)$ is similar to this. The required delay inequalities for

$$V = x_1^2 + x_2^2 + Kz_1^2 + kz_2^2$$

are

$$|x_i(s)| \quad \text{and} \quad |\sqrt{K}\, z_i(s)|$$

$$< \frac{1}{\sqrt{q}} \left[|x_1| + |x_2| + \sqrt{K}\, (|z_1| + |z_2|) \right] .$$

An inspection of Eq. (5.2) reveals that if the method is applied, the coefficients of the negative quadratic terms in x decrease while the coefficient of the cross product x_1x_2 increases. Therefore, a subsequent application of Sylvester's theorem shows that $\dot{V}$ is not even negative-semidefinite. This is a result of the great restrictiveness of the method and, in addition, of the nature of the system (5.1). The fact that $\dot{V}$ is only semidefinite for the linearized system would result in failure even if the very slightest restrictive measure were imposed.

Another fourth-order case is that in which the prompt temperature effect is assumed but the delayed neutrons are not neglected. In this case, the equations are from Eq. (2.13),

$$
\begin{aligned}
x_1 &= -\frac{\beta}{\ell}(k_p+k_c+1)x_1 + \frac{\beta}{\ell}y_1 + \frac{\beta}{\ell}k_c x_2(t-T) - \frac{\beta}{\ell}k_p x_1^2 \\
\dot{y}_1 &= \lambda(x_1-y_1) \\
\dot{x}_2 &= -\frac{\beta}{\ell}(k_p+k_c+1)x_2 + \frac{\beta}{\ell}y_2 + \frac{\beta}{\ell}k_c x_1(t-T) - k_p x_1^2 \\
\dot{y}_2 &= \lambda(x_2-y_2) \quad .
\end{aligned}
\tag{5.3}
$$

If V is

$$V = x_1^2 + x_2^2 + Ky_1^2 + Ky_2^2 \ ,$$

$\dot{V}$ is, if $K\lambda = \beta/\ell$, for the linearized system with no delay,

$$
\begin{aligned}
\frac{1}{2}\frac{\ell}{\beta}\dot{V} = &-(k_p+k_c+1)(x_1^2+x_2^2)-(y_1^2+y_2^2) \\
&+ 2k_c x_1 x_2 + 2(x_1 y_1 + x_2 y_2) \quad .
\end{aligned}
\tag{5.4}
$$

Sylvester's inequalities for Eq. (5.4) are:

(a) $k_p + k_c + 1 > 0$

(b) $k_p + k_c > 0$

(c) $k_p(k_p + 2k_c) + (k_p + k_c) > 0$

(d) $k_p(1 + 2k_c) > 0$.

The inequalities are satisfied for all k_p and $k_c > 0$, hence $\dot{V}$ is negative-definite and the linearized system with no delay is asymptotically stable.

If the delayed neutrons are added to the system (5.1), however, the same problem arises. $\dot{V}$ is only semi-definite, so there is no hope of proving stability for any delay, however small. The effect of the delayed neutrons on the linearized system (5.3) can be analyzed for the case with delay. Equation (5.4) leads to the inequality

$$\frac{1}{2}\frac{\ell}{\beta}\dot{V} \leq -(k_p+1)(x_1^2+x_2^2) + 2(x_1y_1+x_2y_2)$$

$$-(y_1^2+y_2^2) + 2k_c\,|x_1x_2|$$

$$+\sqrt{\frac{\beta}{\lambda\ell}}\,k_c\left[|x_1y_1| + |x_2y_2| + |x_1y_2| + |x_2y_1|\right]$$

for the case independent of delay. If $k_c = 0$, the right side is negative-definite; hence, there should be a k_c sufficiently small for which the linear system can be proved stable independent of delay. The results are summarized in Fig. 5.1. It is noted that the range of allowable parameters for stability is decreased from the case where there are no delayed neutrons. This is due again to the restrictive nature of the method.

Stability of the Sixth Order System

The region of stability is found for the case $T = 0$ by the concentional approach of adding the nonlinear terms to $\dot{V}$ for the linearized system. The region $\dot{V} < 0$ is found mostly by trial and error, as

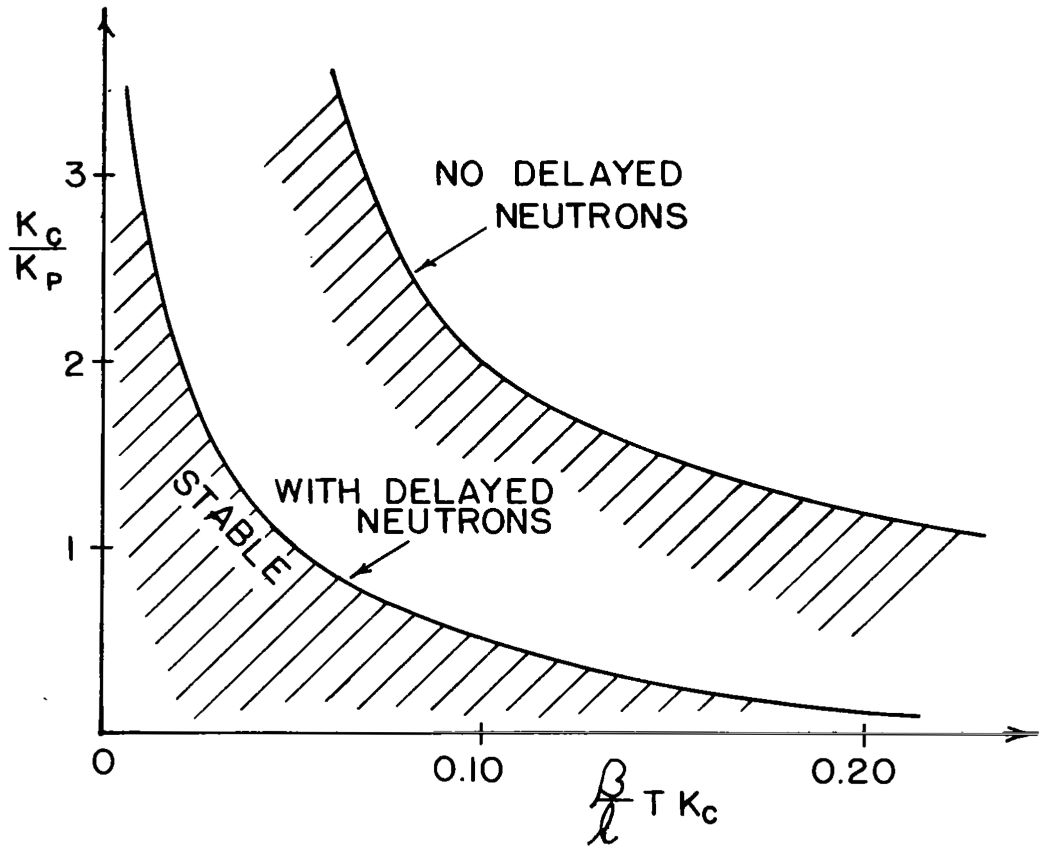


FIGURE 5.1.

EFFECT OF DELAYED NEUTRONS ON LINEAR STABILITY mentioned previously. An estimate is made based upon the region in each plane, then values of the state variables are substituted into $\dot{V}$ for the assumed maximum V. The radius of the region as a function of k_c is shown in Fig. (5.2).

As a comparison to the results, some actual system responses are shown in the last figures in this chapter. Solutions are obtained for $\pm$ 100¢ temperature perturbations in one core. x_1 and x_2 are the power responses for the perturbed and unperturbed core and z_1 and z_2 are the associated temperature responses. The effect of an unrealistically large delay for $k_c = 1$ is also shown. The numbers used therefore exceed the limits of those found mathematically by a large degree. The fact that the system remains stable illustrates that the stability results are extremely conservative. The values $k_p = 1$ and $\frac{\beta}{\ell} = 100$ are used in the simulation studies.

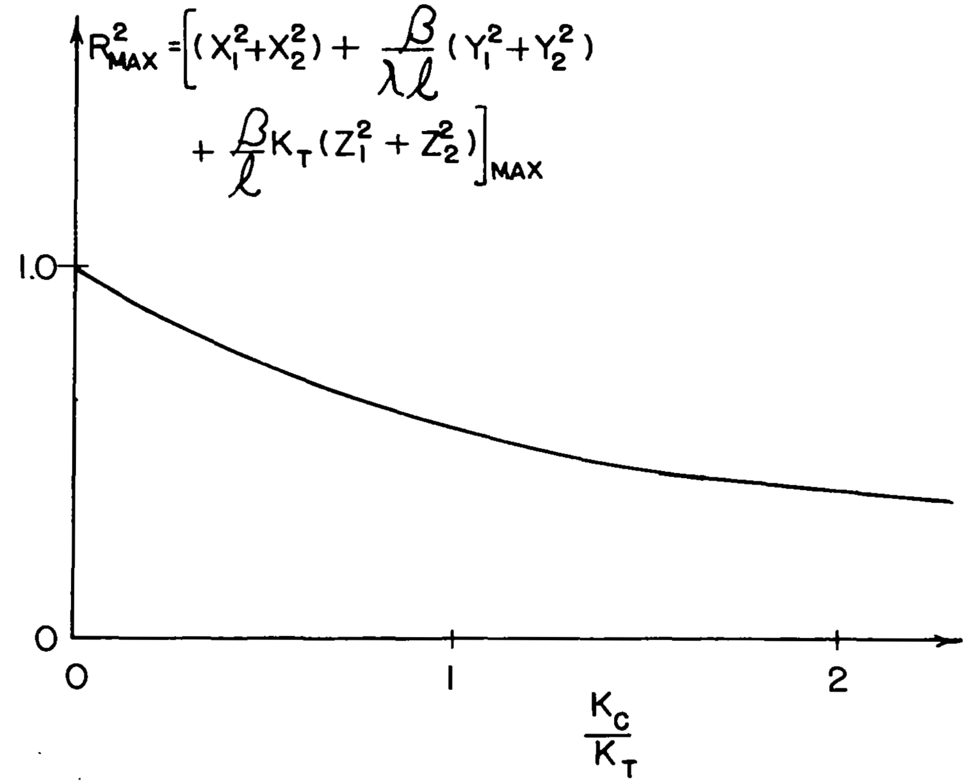


FIGURE 5.2.

NONLINEAR STABILITY OF A

TWO CORE SYSTEM

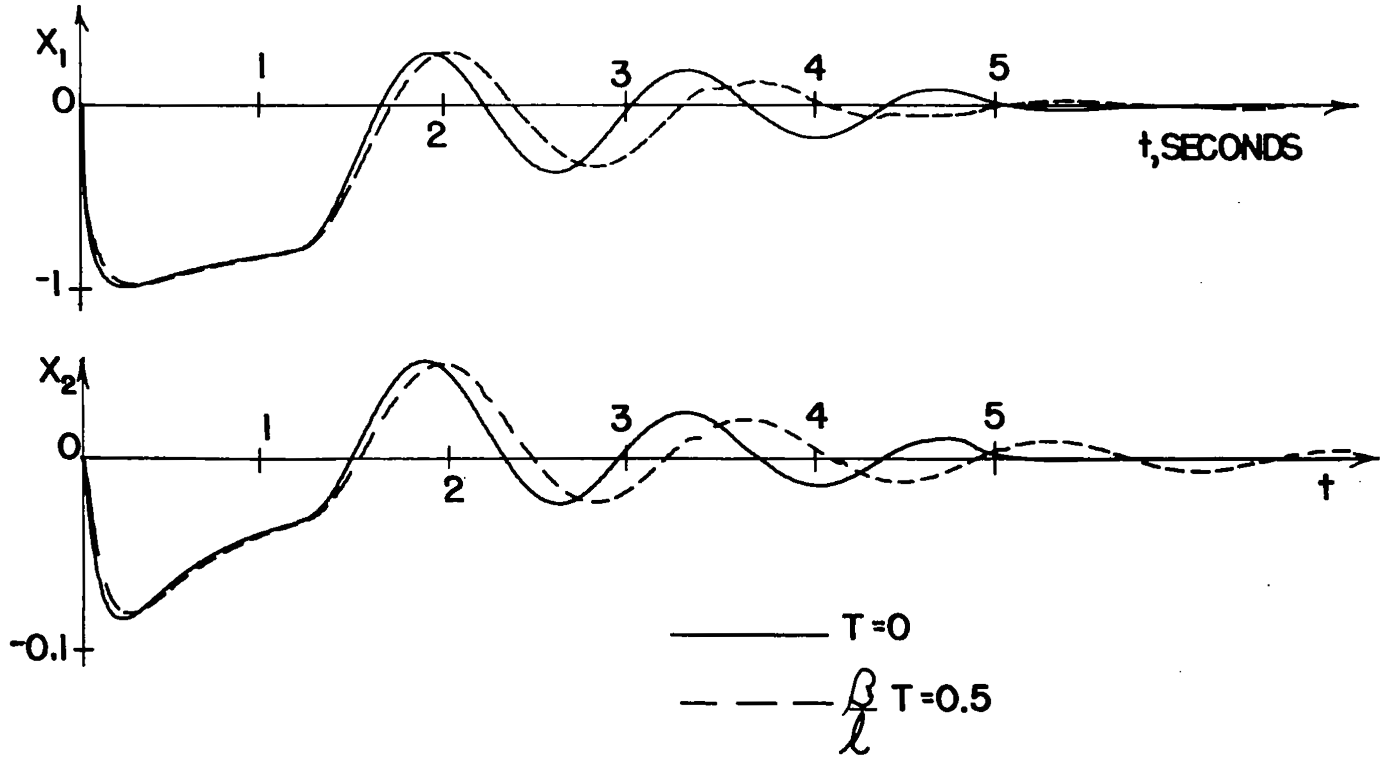


FIGURE 5.3.

POWER RESPONSE, +100% TEMPERATURE DISTURBANCE

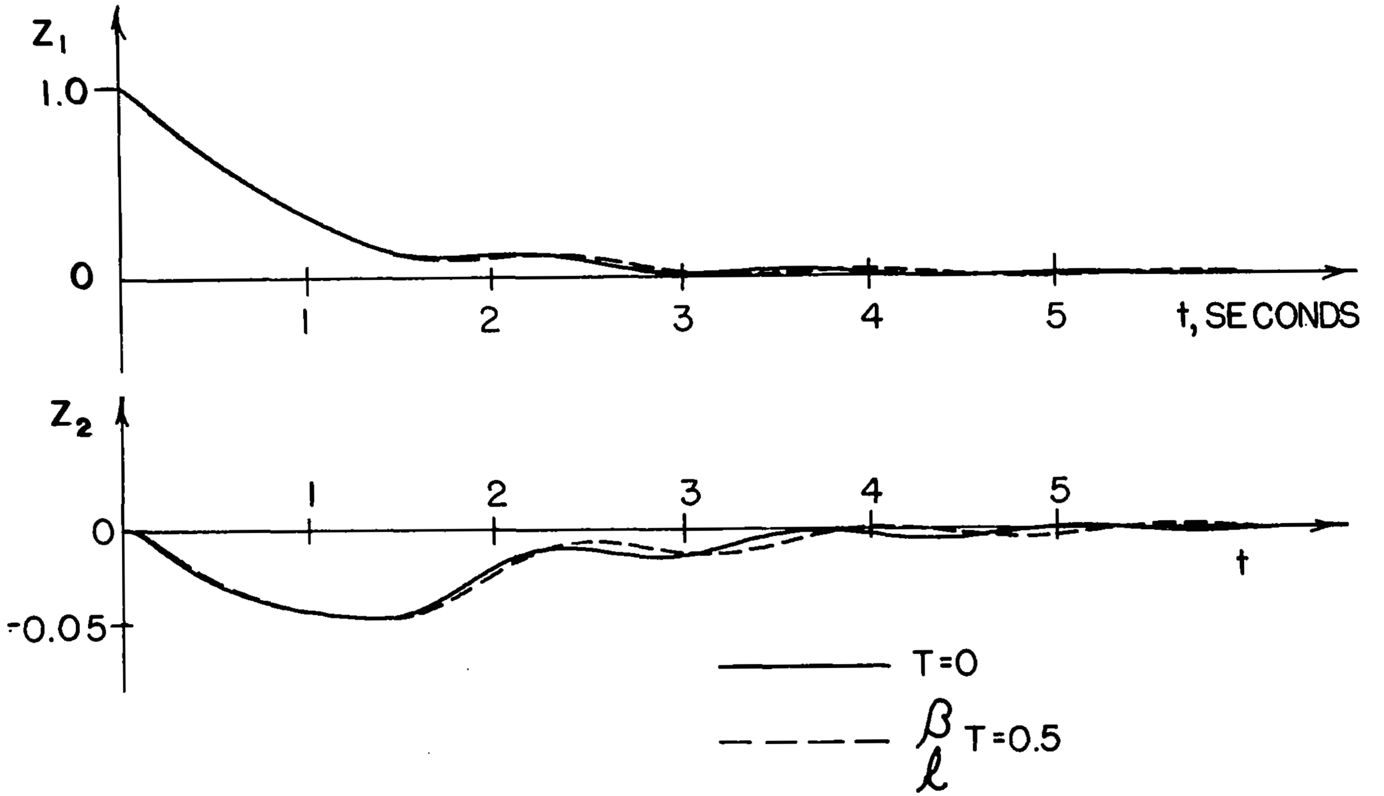


FIGURE 5.4.

TEMPERATURE RESPONSE, +100% TEMPERATURE DISTURBANCE

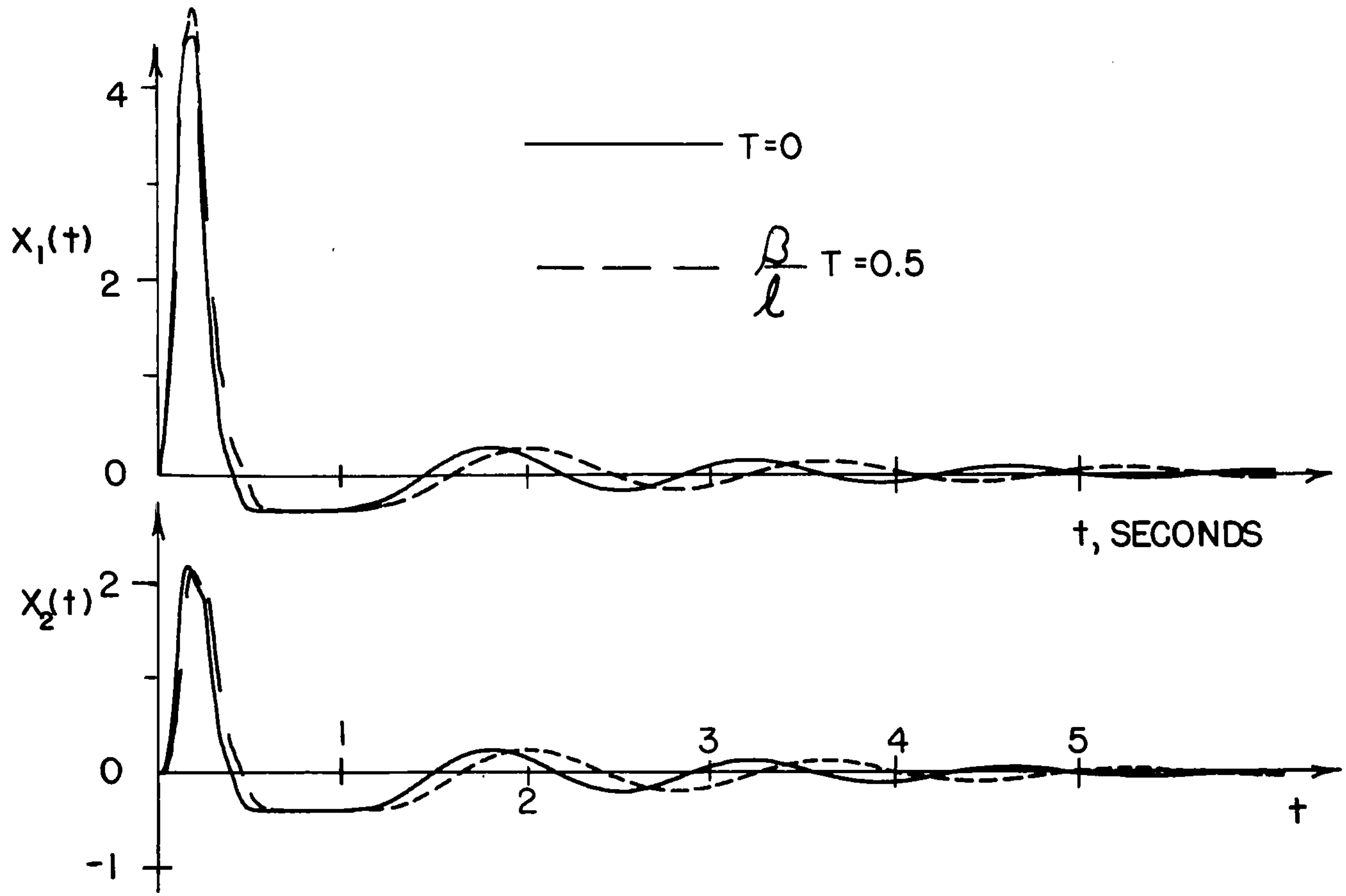


FIGURE 5.5.

POWER RESPONSE, – 100% TEMPERATURE DISTURBANCE

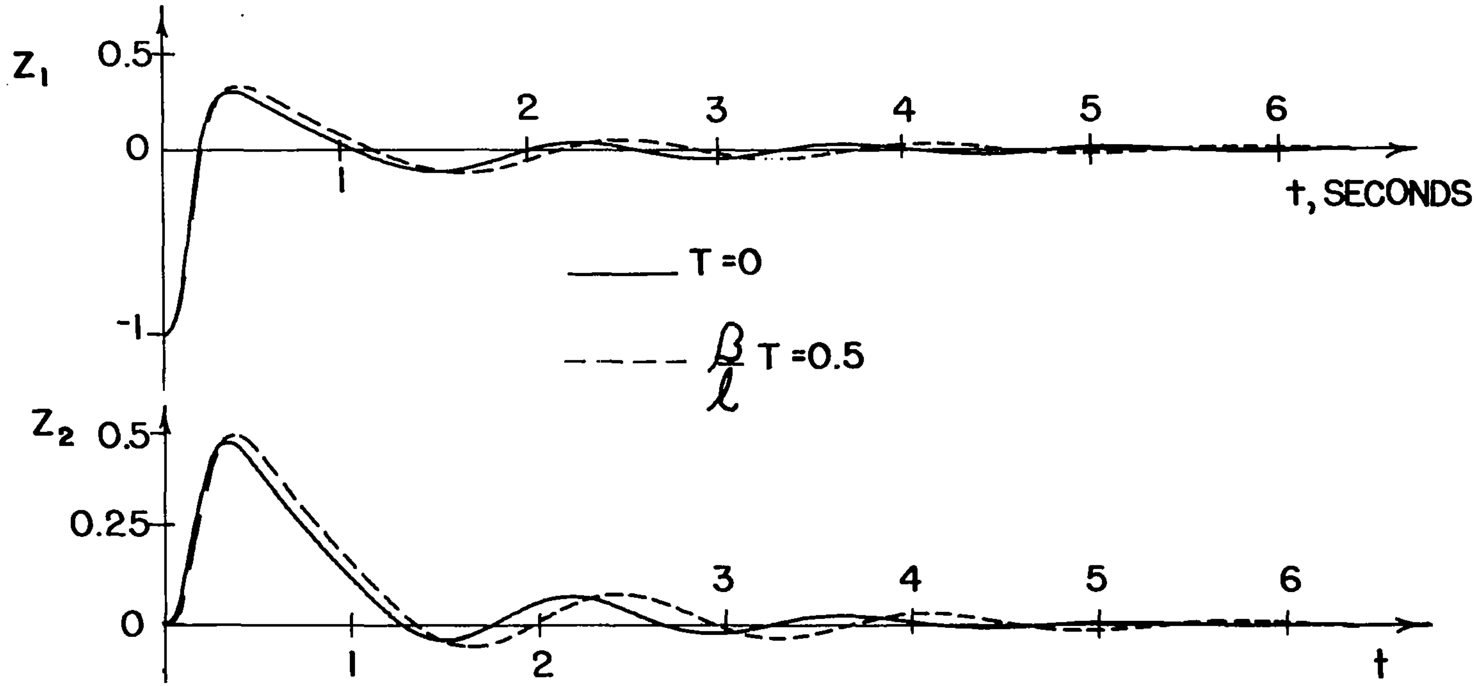


FIGURE 5.6.

TEMPERATURE RESPONSE, -100% TEMPERATURE DISTURBANCE

Chapter 6

CONCLUSIONS

Evaluation of the Results

The results indicate, from a practical viewpoint, the gap between the low order, simplified example and the realistic problem. The results obtained for the second order case in Chapter 4 are quite adequate. However, the results for the realistic, sixth order system are marginal with respect to actual system parameters. The study of the response of the system due to temperature perturbations reveals, moreover, that stability exists for conditions much more severe than would ever be encountered physically. A $\pm$ 100% temperature perturbation, for example, would certainly lead to the destruction of the reactor core at high initial power and temperature. Although the regions of stability may then be adequate, the values of the delay time and coupling coefficient obtained mathematically for linear stability are close to actual parameters. The method is then just barely adequate to prove stability for a realistic system and there is no doubt that the method would fail for an even higher order system than considered here.

The regions of stability obtained for the systems with zero delay are good in light of practical considerations. Again, only fairly small perturbations of the variables are to be expected in the actual system due to material limitations. Also, from the results of the second order problem, the combination of the coupling effect and delay time from the numerical example result in an essentially zero delay case. As the coupling increases, the effect of the delay is quite pronounced; but the magnitude of the coupling in a nuclear rocket cluster, for example, would be limited. The proximity of the cores would be fixed by the associated system equipment such as the nozzles. An important conclusion which can be drawn from the second order study is that the delay time can be neglected if the coupling is less than about one-half of the intrinsic reactivity contribution.at operating conditions.

The fundamental practical conclusion is that there are no stability problems for a system of clustered power reactors, if the system is adequately described by the model used in this study. This conclusion can be extended to systems of several cores since the basic effect of the coupling upon the stability of the system has been determined here.

While there are no stability problems in the mathematical sense the actual response of the system may be somewhat undesirable. This conclusion cannot arise from the mathematical stability analysis; however, the simulation study reveals the oscillatory behavior of the system. This would indicate the desirability of using some form of closed loop control on the system.

Recommendations for Further Study

The answer to the stability question for coupled-core nuclear reactor systems is provided. There are, of course, a great many problems which could be worked for coupled-core systems. The question of automatic control is an interesting one. This is not a stability problem, but due to the pecularity of the coupled system on examination of the proper practical control technique should be made. Each core is essentially an entire system. It must be decided whether to control each core individually or to control just one core and rely upon the inherent stability of the system to provide a desirable behavior.

An application of the approach given in this study might be that of system reduction for stability studies. Probably the most undesirable feature a stability study can have is that there are too many system equations. If the time delay can be handled, however, an approximate reformulation of the model can be made. For example, the temperature-induced reactivity could be written in terms of a true discrete time delay rather than in the form of another equation describing the lag of the temperature behind the power. If there are no delayed neutrons and for a two core system, the equations are

$$\dot{x}_1 = -\frac{\beta}{\ell} k_t x_1(t-\frac{1}{\omega})\left[1+x_1\right] - \frac{\beta}{\ell} k_c(x_1-x_2)$$

$$\dot{x}_2 = -\frac{\beta}{\ell} k_t x_2(t-\frac{1}{\omega})\left[1+x_2\right] - \frac{\beta}{\ell} k_c(x_1-x_2)$$

where the coupling delay is neglected and the reactivity effect is

$$-\frac{\beta}{\ell} k\ x(t-\frac{1}{\omega}) \quad .$$

The delay time $\frac{1}{\omega}$ is the mean time which the temperature lags the power from Eq. (2.13c). A linear time delay stability analysis on this problem shows that, for stability,

$$\omega > 2 \frac{\beta}{\ell} (k_t + 2k_c).$$

If $k_t = 1$, $\frac{\beta}{\ell} = 100$, and $k_c = 0.1$,

$$\omega > 240 \quad .$$

For the numbers given here, ω is actually about unity. This approach does not seem to be particularly interesting for this problem.

However, there is a tendency to neglect the delayed neutrons in kinetics studies. This may not be a bad assumption in stability studies, but prompt neutron kinetics are inadequate for more detailed dynamics studies. The delayed neutron effect can then be approximated by

$$- \frac{\beta}{\ell} x + \frac{\beta}{\ell} x(t - \frac{1}{\lambda})$$

instead of

$$- \frac{\beta}{\ell} x + \frac{\beta}{\ell} y$$

plus another differential equation for y(t). It is not known if these representations provide sufficiently accurate approximations.

An area for significant advancement is in that of the purely mathematical features of this problem. The method as presented here is clearly restricted to systems which can be easily linearized, are of low order, and for which a quadratic V function can be written.

The use of a non-quadratic V would require that the f(V) be different from V/q. Some work was done on this. For example, in a single reactor with no delayed neutrons a V function of the form

$$V = x - \ln(1 + x) + Kz^2$$

can be found from the Variable Gradient Method (Schultz, 1962). This would suggest an f(V) of the general form

$$f(V) \sim e^V \quad ;$$

however, no suitable form could be found. In a coupled system, the logarithmic V does not result. The form is quadratic with quadratic coefficients. A quadratic f(V) is unsuitable because, for example, if

$$F(V) = V^2 \quad ,$$

the condition $f(V) > V$ is not always satisfied.

It is essential that these questions be explored further, for there are many real problems in which the concept of a time lag is involved. In nuclear reactor systems control there could be lags in the mechanical portion of the control rod drive mechanism. The method presented here, however, succeeds in providing basic conclusions on coupled-core stability, and leads to good results for the low order cases.

REFERENCES

Driver, Rodney D. "Existence and Stability of Solutions of a Delay-Differential System," Archive for Rational Mechanics and Analysis, 16 (1962), pp. 131-133.

Driver, Rodney D. Personal Communication. The Sandia Corporation, Albuquerque, New Mexico (1965).

Krasovskii, N. N. Stability of Motion. Stanford University Press: Stanford, California (1963).

La Salle, Joseph and Lefschetz, Solomon. Stability by Liapunov's Direct Method with Applications. The RAND Corporation, Mathematics in Science and Engineering Series, 4: Academic Press (1961).

Mohler, Ronald R. "Optional Control of Nuclear Reactor Processes," Doctoral Dissertation, University of Michigan: Ann Arbor (1965).

Razumikhin, B. S. "The Application of Lyapunov's Method to Problems in the Stability of Systems with Delay," Automation and Remote Control, 21 (1960), pp. 515-520.

Schultz, Donald G. "The Variable Gradient Method of Generating Liapunov Functions with Applications to Automatic Control Systems," Doctoral Dissertation, School of Electrical Engineering, Purdue University: Lafayette, Indiana (1962).

Seale, Robert L. "Coupled Core Reactors," Los Alamos Scientific Laboratory Report No. LAMS-2967 (1964a).

Seale, Robert L. "Investigation of Coupled ROVER Cores," Los Alamos Scientific Laboratory Report No. LAMS-2968 (1964b).

Weaver, Lynn E. System Analysis of Nuclear Reactor Dynamics. American Nuclear Society and United States Atomic Energy Commission Monograph Series on Nuclear Science and Technology, Rowman and Littlefield: New York (1963).

Appendix A

ANALOG SIMULATION OF THE TIME DELAY

A delay circuit is constructed on a Computer Systems Inc. Model 5800 repetitive analog computer by using the memory feature of the integrators. In this mode, a control pulse is applied to the integrator, causing it to reset and operate with the frequency of the control pulse. The integrator follows the function in the reset mode, then holds the final value during the operate period if the function is applied to the initial condition input of the amplifier. If a reverse pulse is applied, the initial value of the input is held during reset, and the function is tracked during operate.

Figure (A-1) is the basic circuit with four optional outputs, depending upon the type of output desired. The symbol M indicates a normal control pulse, R the reverse pulse, and no symbol indicates that the integrator is operating in its normal mode.

For a general function, the outputs of the four memory units are shown in Figure (A-2). A step approximation to $x(t-T)$ can be obtained from #2, #3, or #4, with the delay time

$$T = (\text{ No. of Amplifiers } - 1)/2\omega,$$

where ω is the frequency of the control pulse. Also, a straight line approximation can be constructed using a real time integrator. Note that the reset and operate cycles of #2 and #4 are in phase, but #2 leads #4 in time by one cycle. The outputs can then be used as stored values

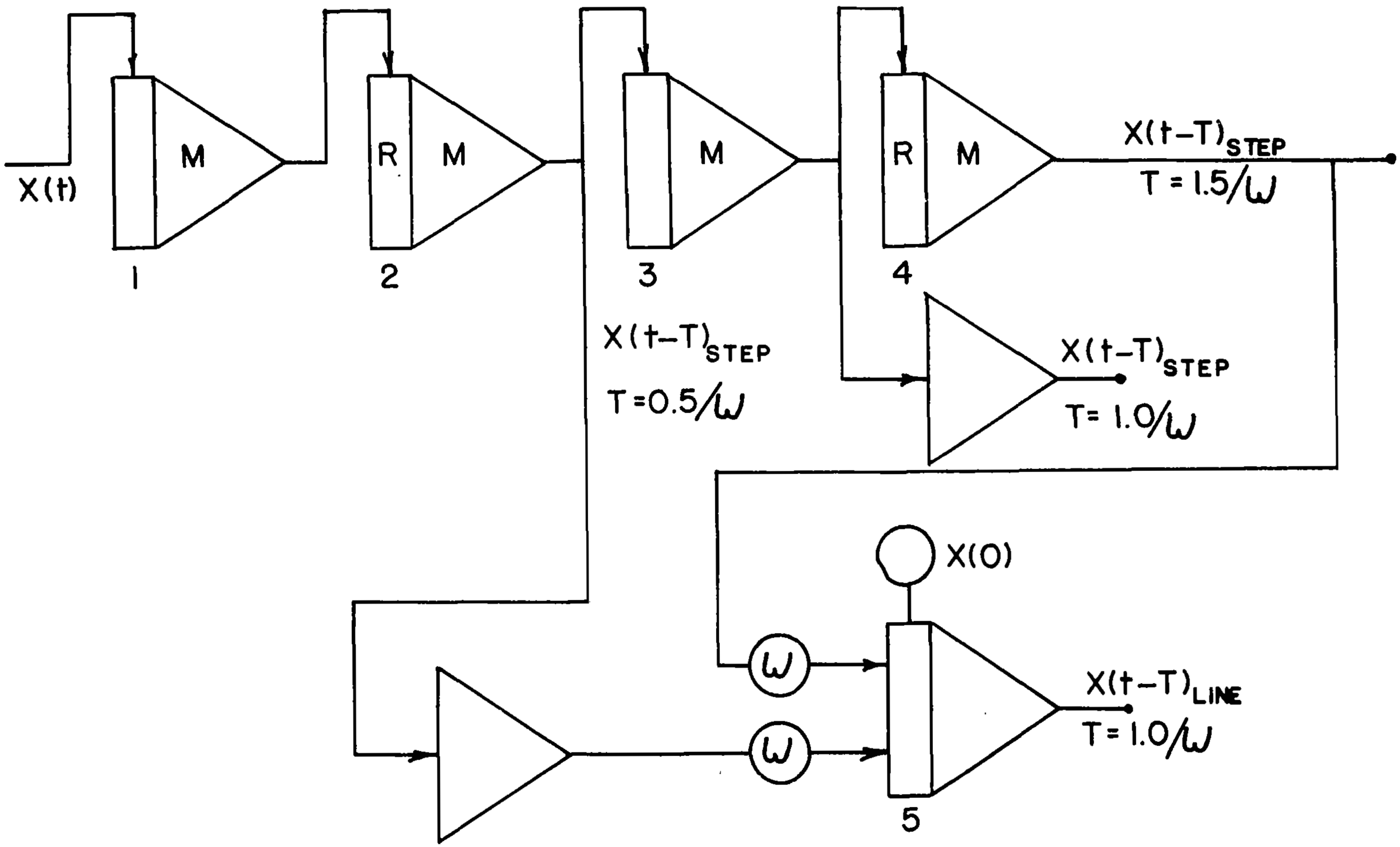


FIGURE A.I. TIME DELAY CIRCUIT

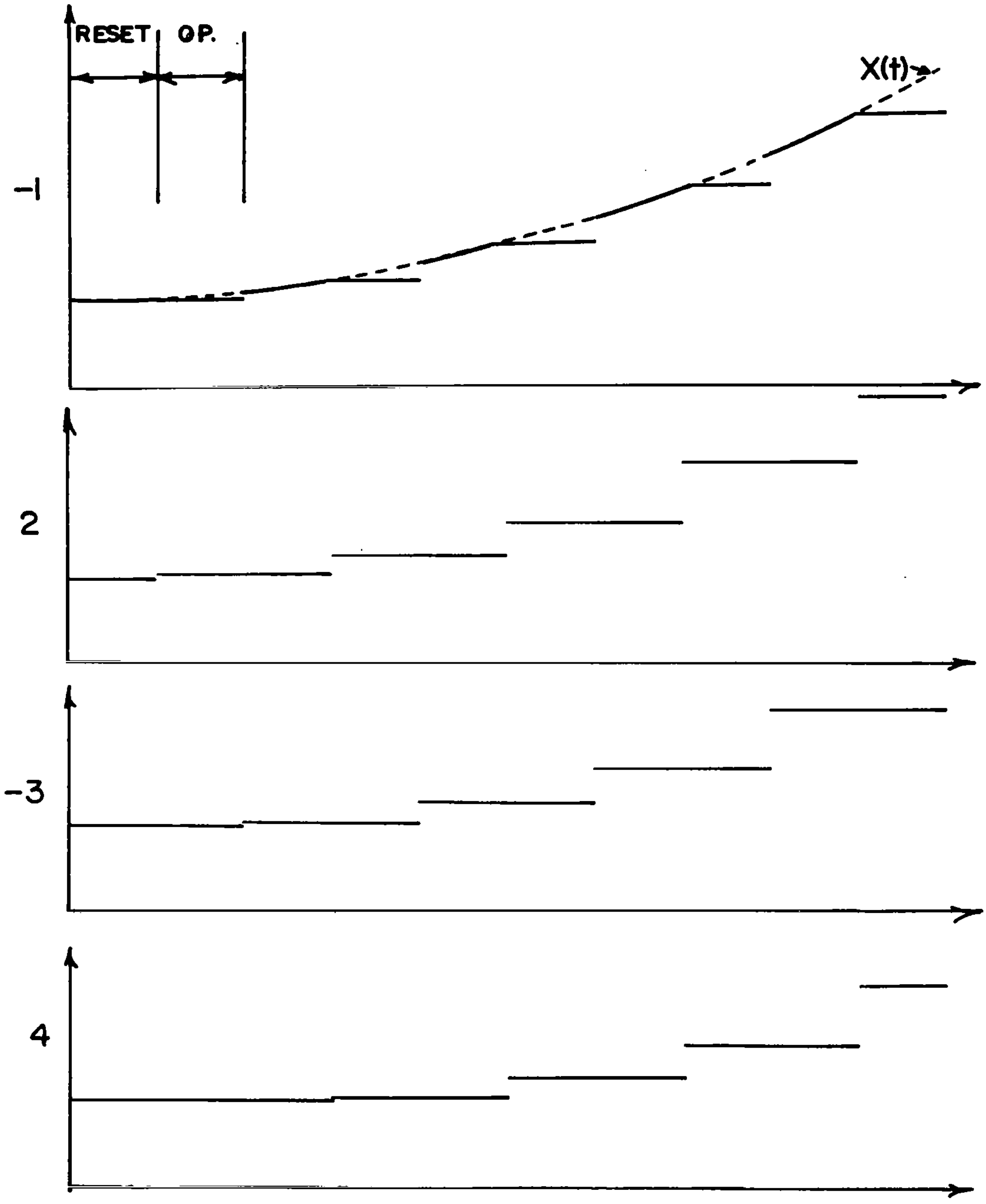


FIGURE A.2.

MEMORY INTEGRATOR OUTPUTS

sampled from $x(t)$ at times $1/\omega$ apart. Therefore, the straight line can be constructed as follows:

$$x(t-\frac{1}{\omega}) = x_0 + \int \omega(x_2 - x_4)dt$$

where $\omega(x_2-x_4)$ is the slope, a constant, over each cycle. x_0 is the initial value of $x(t)$.

It is found that in actual use, the straight line approximation is quite difficult to use. For short delay times, it is difficult to reproduce properly delayed functions. The results are extremely sensitive to variations in the potentiometer settings for ω. However, for delay times or less than about 0.5 seconds, the step function provides a sufficiently accurate solution.

NASA-Langley, 1966 CR-447